Graded structures of classical field theory

XIX Young Researchers Workshop in Geometry Dynamics and Field Theory

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Structure of the talk

Poisson an Dirac structures in Classical Mechanics

Graded Poisson and Dirac structures

Foliations

Poisson an Dirac structures in Classical Mechanics

Poisson structures

Definition
A Poisson structure on a smooth manifold M is a bracket $\{\cdot,\cdot\}$ defined on $C^\infty(M)$ that satisfies:

- Skew-symmetry, $\{f,g\} = -\{g,f\}$
- Leibniz identity, $\{fh,g\} = f\{h,g\} + h\{f,g\}$
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Going degenerate: Dirac structures

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A Dirac structure on a smooth manifold M is a bracket $\{\cdot,\cdot\}$ defined on

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for every U open subset, where K is a smooth integrable distribution on M (maybe not of constant rank) that satisfies

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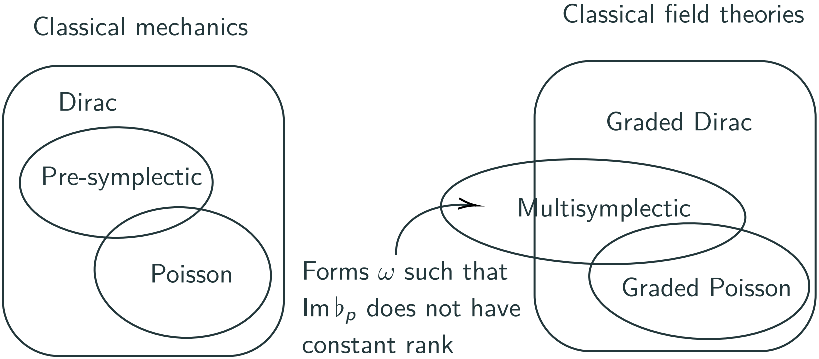
Objective

Translate this picture to classical field theories.

More particularly,

Find an equivalence between brackets and some tensorial object.



Graded Poisson and Dirac structures

Multisymplectic manifolds

In order to work with any of the formalisms (Lagrangian, extended Hamiltonian, reduced Hamiltonian), we will work abstractly with a multisymplectic manifold, (M, ω), a manifold M together with a closed $(n+1)-$form.

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We have already a lot of examples:

- $(J^1\pi, \Omega_{\mathcal{L}})$, where $\mathcal{L}$ is a Lagrangian;
- $(\bigwedge_2^n Y, \Omega)$;
- (Z^*, Ω_h), where $h : Z^* \to \bigwedge_2^n Y$ is a Hamiltonian section;
- (M, ω), where M is an orientable manifold and ω is a volume form.

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where $\alpha \in \Omega^{n-q}(M)$ is called the corresponding Hamiltonian form.

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We have the following:

Proposition
If U, V are Hamiltonian multivector fields of degree p, q. Then $[U, V]$ is a Hamiltonian multivector field of order $p + q - 1$. The corresponding Hamiltonian form is

$$(-1)^q \iota_{U \wedge V} \Omega.$$

Graded Poisson brackets

The previous proposition induces the following:

Definition
Let α, β be Hamiltonian forms of order $k - p, k - q$, respectively. Define their Poisson bracket

$$\{\alpha, \beta\} := (-1)^q \iota_{U \wedge V} \Omega,$$

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Then, we have

Theorem
Modulo exact forms, the previous brackets defines a graded Lie algebra on the space of Hamiltonian forms

Question: Does this recover the multisymplectic form?

Properties of graded Poisson brackets

If we set $\deg \beta := k -$ order of β, then th Poisson bracket satisfies:

- *It is graded:*

$$\deg\{\alpha, \beta\} = \deg \alpha + \deg \beta;$$

- *It is graded-skew-symmetric:*

$$\{\alpha, \beta\} = -(-1)^{\deg \alpha \deg \beta}\{\beta, \alpha\};$$

- *It is local:* If $d\alpha|_x = 0$, $\{\alpha, \beta\}|_x = 0$
- *It satisfies graded Jacobi identity (up to an exact term):*

$$(-1)^{\deg \alpha \deg \gamma}\{\{\alpha, \beta\}, \gamma\} + \text{cyclic terms} = \text{exact form.}$$

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- *It satisfies Leibniz identity:* For $a = k$, if $\beta \wedge d\gamma \in \Omega_H^{b+c-1}(M)$, then

$$\{\beta \wedge d\gamma, \alpha\} = \{\beta, \alpha\} \wedge d\gamma + (-1)^{k-\deg \beta} d\beta \wedge \{\gamma, \alpha\};$$

- *It is invariant by symmetries:* If $X \in \mathfrak{X}(M)$ and $\pounds_X \alpha = 0$, then $\iota_X \alpha \in \Omega_H^{a-2}(M)$ and

$$\{\iota_X \alpha, \beta\} = (-1)^{\deg \beta} \iota_X \{\alpha, \beta\};$$

Onto the definitions...

First, we look at the linearized version:

Definition
Let M be a manifold. A graded Dirac structure of order n is a tuple $(S^a, K_p, \sharp_a)$, where $S^a \subseteq \bigwedge^a M$ is a vector subbundle of forms, $K_p \subseteq \bigvee_p M (= \bigwedge^p TM)$ is a subbundle of multivectors, and

$$\sharp_a : S^a \to \bigvee_{n+1-a} M / K_{n+1-a}$$

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are linear bundle maps sastifying:

- $K_p = (S^a)^{\circ,p}$, for $p \leq a$.
- The maps $\sharp_a$ are *skew-symmetric*, that is,

  $$\iota_{\sharp_a(\alpha)}\beta = (-1)^{(n+1-a)(n+1-b)}\iota_{\sharp_b(\beta)}\alpha,$$

  for all $\alpha \in S^a$, $\beta \in S^b$.

And it is integrable:

- It is *integrable:* For $\alpha : M \to S^a$, $\beta : M \to S^b$ sections such that $a + b \leq 2n + 1$, and U, V multivectors of order $p = n + 1 - a$, $q = n + 1 - b$, respectively such that

$$\sharp_a(\alpha) = U + K_p, \ \sharp_b(\beta) = V + K_q,$$

we have that the $(a + b - k)$-form

$$\theta := (-1)^{(p-1)q} \pounds_U \beta + (-1)^q \pounds_V \alpha - \frac{(-1)^q}{2} d \left(\iota_V \alpha + (-1)^{pq} \iota_U \beta \right)$$

takes values in S_{a+b-k}, and

$$\sharp_{a+b-k}(\theta) = [U, V] + K_{p+q-1}.$$

Field theories as an example

Let (M, ω) be a multisymplectic manifold of order n, that is, the form ω must satisfy $\iota_v \omega = 0$ if and only if $v = 0$. Then, the (linear) correspondence between Hamiltonian multivector fields and forms defines a graded Dirac structure of order n:

Field theories as an example

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- $$S^a = \{\iota_U \omega : U \in \bigvee_{n+1-a} M\};$$
- $$K_p = \ker_p \omega;$$
- $\sharp_a : S^a \to \bigvee_{n+1-a} / K_{n+1-a}$ is given by

$$\sharp_a(\alpha) = U + K_{n+1-a} \text{ if and only if } \iota_U \omega = \alpha.$$

In this case, $\sharp_a$ are the inverse of the $\flat_p$ (contraction) maps induced by ω.

Relationship with graded Poisson brackets

Given a graded Dirac structure on M, $(S^a, \sharp_a, K_p)$, we can define a Hamiltonian form as an $(a-1)-$form, α such that $d\alpha \in S^a$.

Definition
The Poisson bracket of Hamiltonian forms is given by

$$\{\alpha, \beta\} := (-1)^{\deg \beta} \iota_{\sharp_b(d\beta)} d\alpha$$

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Theorem
Under some integrability conditions on the sequence of subspaces S^a, any graded Dirac structure on this family is completely characterized by the graded Poisson bracket it induces. That is, we get a 1-1 correspondence

$$\{\textit{Graded Poisson structures}\} \cong \{\textit{Graded Poisson brackets}\}.$$

But why?

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- The quest of finding a bracket formulation of field theories.

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- Tools.

	Symplectic	Poisson	Dirac
Easily restricted?	Yes	No	Yes
Easily Quotiented?	No	Yes	Yes

	Multisymplectic	Graded Poisson	Graded Dirac
Easily restricted?	Yes	No	Yes
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Integrability conditions

What are the integrability conditions on the sequence S^a?

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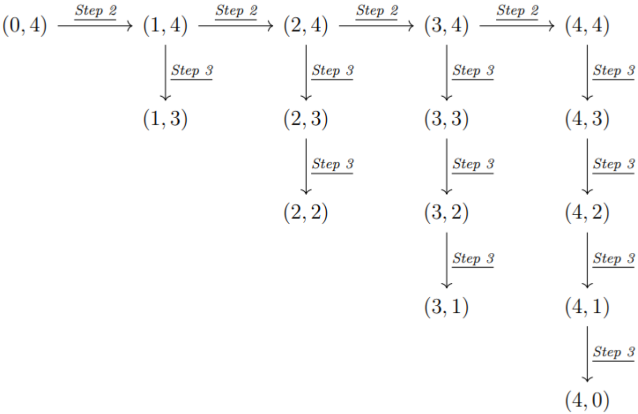
$$S^b = \langle df_i^j \wedge d\gamma_{ij}, i \rangle;$$

- For each $1 \leq a \leq k$, locally, there exists a family of Hamiltonian forms forms γ^j, and a family of vector fields X^j such that
$$S^a = \langle d\gamma^j \rangle \; \pounds_{X^j}\gamma^j = 0,$$
and
$$S^{a-1} = \langle d\iota_{X^j}\gamma^j \rangle.$$

Idea of the Proof



Foliations

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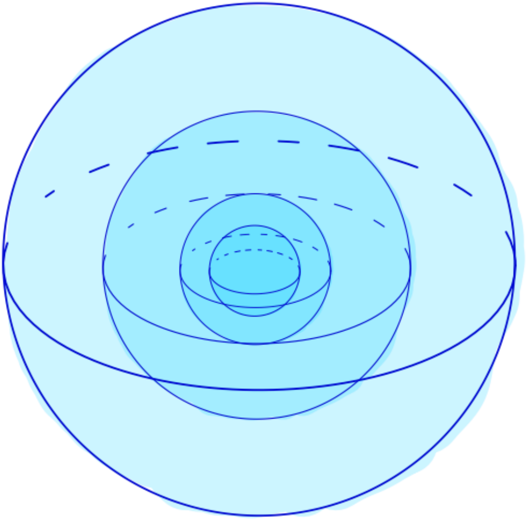
- Graded Dirac structures $\implies$ Multisymplectic (possibly degenerate) foliations.
- Multisymplectic foliations (with some technical conditions) $\implies$ Graded Dirac structures.
- The correspondences are not inverse of the other!

Are there non-trivial examples? Yes!



Final remarks and future research

- We developed the theory of Poisson bracket and tensors in classical field theories;
- Can we find an analogue to Lie-Poisson structures in this setting?;
- How are these bracket and structures related to reduction?

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Thank you for your attention!