Geometría Multisimpléctica, el marco para Teorías Clásicas de Campos

Seminario de doctorandos

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ICMAT-UCM

Structure of the talk

The first jet bundle

Objective: Look for an intrinsic formulation of Classical Field Theories (or calculus of variations)

Tool needed: A bundle that factorizes finite order differential operators

The first jet bundle I

Given a fibre bundle (fibered manifolds also work)

$$\pi : Y \to X,$$

we want a bundle

$$J^1\pi \to Y$$

that factorizes first order differential operators defined on sections

$$\phi : X \to Y.$$

Definition (First order differential operator)
A first order differential operator is a map:

$$D : \{\text{Sections of } Y \xrightarrow{\pi} X\} \to \{\text{Sections of } Z \xrightarrow{\tau} X\},$$

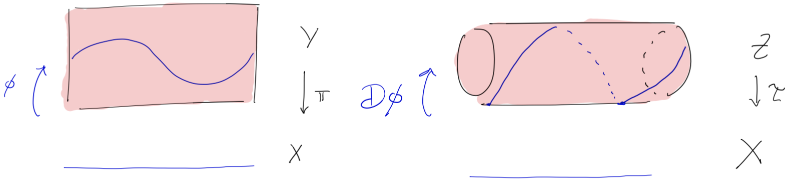
$$x = \tau\left(D\phi(x)\right).$$

such that $D\phi(x) = D\psi(x)$ whenever

$$\phi(x) = \psi(x),\ d_x\phi = d_x\psi$$

The first jet bundle II

In other (more understandable) words, we have an operation on sections, where the value of $D\phi$ on $x \in X$ only depends on x, $\phi(x)$, and $d_x\phi$:



$$(D\phi)(x) = \widetilde{D}(x, \phi(x), d_x\phi)$$

Problem: Find the manifold where $\widetilde{D}$ is defined.

The first jet bundle III

Definition (The first jet bundle)
Given a fibre bundle (or fibered manifold) $Y \xrightarrow{\pi} X$, define the first jet bundle (as a set) as

$$J^1\pi := \{d_x\phi, \text{where } \phi : U \to Y\big|_U \text{ is a local section}\}$$

Theorem
$J^1\pi$ can be endowed whith a smooth manifold structure with the coordinates

$$(x^\mu, y^i, z^i_\mu),$$

where z^i_μ represents $\frac{\partial \phi}{\partial x^\mu}$, and

$$\pi(x^\mu, y^i) = x^\mu$$

are fibered coordinates on $Y \xrightarrow{\pi} X$.

The first jet bundle IV

Then, given a first order differential operator

$$D : \{\text{Sections of } Y \xrightarrow{\pi} X\} \to \{\text{Sections of } Z \xrightarrow{\tau} X\},$$

with coordinate expression

$$(D\phi)(x^\mu) = \widetilde{D}\left(x^\mu, \phi^i, \frac{\partial \phi^i}{\partial x^\mu}\right),$$

it induces a bundle map over X

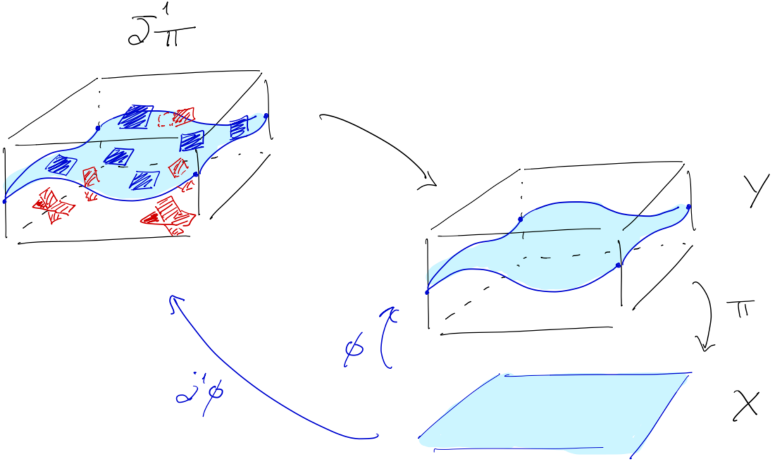
$$\widetilde{D} : J^1\pi \to Z,$$

and

$$D\phi = \widetilde{D} \circ j^1\phi,$$

where $j^1\phi : X \to J^1\pi$ is the first jet lift (a section storing derivatives up to first order).

The first jet bundle V



The geometry of calculus of variations

The geometric setting I

What to minimize/maximize? Sections!

Fixed some fibered manifold

$$Y \xrightarrow{\pi_{YX}} X \text{ with coordinates } (x^\mu, y^i) \mapsto x^\mu,$$

we want to find a section

$$\phi : X \to Y,\ (x^\mu) \mapsto (x^\mu, y^i = \phi^i(x^\mu))$$

minimizing/maximizing the functional

$$\mathcal{J}[\phi] = \int_X L\left(x^\mu, \phi^i(x^\mu), \frac{\partial \phi^i}{\partial x^\mu}, \frac{\partial^2 \phi^i}{\partial x^\mu \partial x^\nu}, \dots\right) d^n x.$$

We will focus on first order theories,

$$\mathcal{J}[\phi] = \int_X L\left(x^\mu, \phi^i(x^\mu), \frac{\partial \phi^i}{\partial x^\mu}\right) d^n x.$$

The geometric setting II

$\mathcal{L}(\phi) = L(x^\mu, \phi^i, \frac{\partial \phi^i}{\partial x^\mu}) d^n x$ is called the Lagrangian density, and can be interpreted as a first order differential operator

$$\mathcal{L} : \{\text{Fields}\} \to \{n\text{-forms on } X\},$$

$$\mathcal{L} : \{\text{Sections of } Y \xrightarrow{\pi} X\} \to \{\text{Sections of } \bigwedge^n X \xrightarrow{\tau} X\}.$$

Therefore, it factorizes through a fibered map (which will be denoted and called the same)

$$\mathcal{L} : J^1\pi \to \bigwedge^n X$$

The geometric setting III

We can interpret

$$L\left(x^{\mu}, \phi^{i}(x^{\mu}), \frac{\partial \phi^{i}}{\partial x^{\mu}}\right) d^{n}x$$

as an *n*-form on the first jet bundle

$$J^1\pi_{YX} \text{ with coordinates } (x^{\mu}, y^{i}, z^{i}_{\mu}).$$

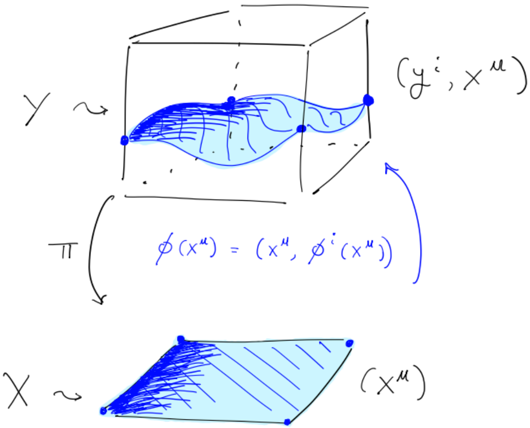
We call it the Lagrangian density

$$\mathcal{L} = L(z^{\mu}, y^{i}, z^{i}_{\mu}) d^{n}x.$$

We can rewrite the action as

$$\mathcal{J}[\phi] = \int_X (j^1\phi)^* \mathcal{L}.$$

The geometric setting IV



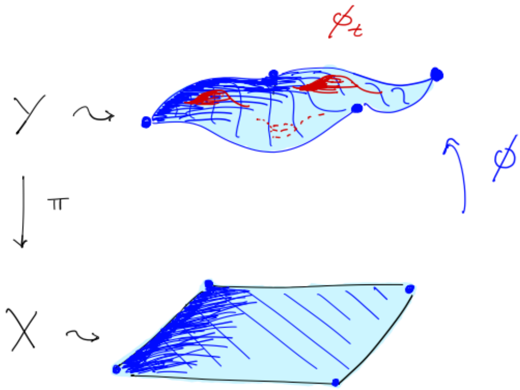
$$\mathcal{J}[\phi] = \int_X (j^1\phi)^*\mathcal{L},$$

where $\mathcal{L} \in \Omega^n(J^1\pi_{YX})$ is the Lagrangian dentisy.

The Euler-Lagrange equations I

If ϕ is a minimizer/maximizer (more generally, stationary section),

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{J}[\phi_t] = 0, \forall \text{ variation } \phi_t.$$



The Euler-Lagrange equations II

Equivalently,

$$0 = \int_X \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (j^1 \phi_t)^* \mathcal{L}.$$

Locally, we get

$$\frac{\partial L}{\partial y^i} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial L}{\partial z^i_\mu}\right).$$

What about intrinsic Euler-Lagrange equations?

If we define

$$\xi := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \phi_t = \xi^i \frac{\partial}{\partial y^i} \in \mathfrak{X}(Y),$$

$$\xi^{(1)} := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} j^1\phi_t = \xi^i \frac{\partial}{\partial y^i} + \left(\frac{\partial \xi^i}{\partial x^\mu} + \frac{\partial x^i}{\partial y^j} z^j_\mu\right)\frac{\partial}{\partial z^j_\mu} \in \mathfrak{X}(J^1\pi_{YX}).$$

The Euler-Lagrange equations III

If we define

$$\xi := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t = \xi^i \frac{\partial}{\partial y^i} \in \mathfrak{X}(Y),$$

$$\xi^{(1)} := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j^1\phi_t = \xi^i \frac{\partial}{\partial y^i} + \left(\frac{\partial \xi^i}{\partial x^\mu} + \frac{\partial x^i}{\partial y^j} z^j_\mu\right) \frac{\partial}{\partial z^j_\mu} \in \mathfrak{X}(J^1\pi_{YX}),$$

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{J}[\phi_t] = \int_X (j^1\phi)^* \pounds_{\xi^{(1)}} \mathcal{L}, \text{ for every vertical } \xi \in \mathfrak{X}(Y)$$

Applying Stokes' Theorem

$$0 = \int_X (j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L} + \int_X d\iota_{\xi^{(1)}} \mathcal{L} = \int_X (j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L}.$$

The Euler-Lagrange equations IV

$$0 = \int_X (j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L} \text{ for every vertical } \xi \in \mathfrak{X}(Y).$$

Does not yield equations.

Idea: modify $\mathcal{L}$

We want to find an n-form $\Theta_{\mathcal{L}}$ satisfying

$$(j^1\phi)^*\mathcal{L} = (j^1\phi)^*\Theta_{\mathcal{L}}$$

such that ϕ is an stationary field of the action if and only if

$$0 = \int_X (j^1\phi)^* \iota_{\eta} d\Theta_{\mathcal{L}} \text{ for every } \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

The Euler-Lagrange equations V

Proposition
There is such $\Theta_{\mathcal{L}}$, and can be intrinsically defined (using the geometry of $J^1\pi_{YX}$).

Locally,

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x_\mu - \left(\frac{\partial L}{\partial z^i_\mu} z^i_\mu - L \right) d^n x$$

and it is called the Poincaré-Cartan form.

Corollary (Intrinsic Euler-Lagrange equations)
A field $\phi : X \to Y$ is stationary if and only if it satifies

$$(j^1\phi)^* \iota_\eta d\Theta_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

Define the multisymplectic form as

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}}.$$

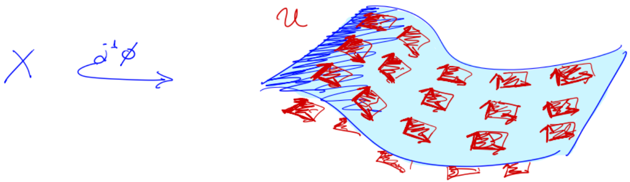
Looking for solutions

To find solutions, we can look for distributions on $J^1\pi_{YX} \to$ such that an integral section of such this distribution $\sigma : X \to J^1\pi_{YX}$ satisfies

$$\sigma^* \iota_\eta \Omega_{\mathcal{L}} = 0, \forall \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

We can define such distributions via decomposable n-multivector fields

$$U = X_1 \wedge \cdots \wedge X_n.$$



Then, being stationary is characterized by $\iota_U \Omega_{\mathcal{L}} = 0$.

Disclaimer

Giving such a multivector field U does not immediately give a solution:

- We need to make sure that the corresponding distribution is integrable.
- Even if it is integrable, it may not be holonomic. That is, that the corresponding integral section $\sigma : X \to J^1\pi_{YX}$ could fail to be the jet lift of some section
$$\phi : X \to Y.$$
When $\mathcal{L}$ is regular, this is not an issue.
- Even if it satisfies the previous conditions, there may not exist global sections of $Y \xrightarrow{\pi_{YX}} X$.

Summary

- Fields, denoted by ϕ, are sections of a fibered manifold $Y \xrightarrow{\pi_{YX}} X$.
- A first order variational problem is defined through a Lagrangian density $\mathcal{L}$ on $J^1\pi_{YX}$ (which defines an n-form on X at each point), and the action can be expressed as

$$\mathcal{J}[\phi] = \int_X (j^1\phi)^* \mathcal{L}.$$

- If we define the multisymplectic form as

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}},$$

stationary fields are characterized by

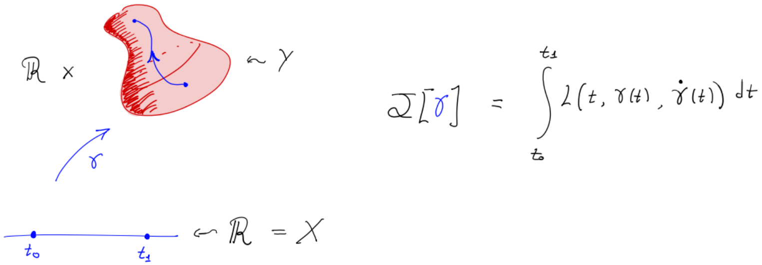
$$(j^1\phi)^* \iota_\eta \Omega_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

In particular, we can look for decomposable horizontal n-multivector fields U satisfying

$$\iota_U \Omega_{\mathcal{L}} = 0.$$

The Hamiltonian formalism

Let Q be a configuration manifold. We can interpret Classical Mechanics as a classical field theory



Then,

$$J^1\pi = TQ \times \mathbb{R}.$$

Hamiltonian formalism in Classical Mechanics II

The Hamiltonian formalism is obtained via the Legendre transformation,

$$TQ \times \mathbb{R} \xrightarrow{\mathrm{Leg}_L} T^*Q \times \mathbb{R} \times \mathbb{R},$$

$$(t, q^i, \dot{q}^i) \mapsto \left(t, q^i, p_i = \frac{\partial L}{\partial \dot{q}^i}, p^1 = -\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i + L \right).$$

The Poincaré-Cartan form in this case is

$$\theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i - \left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L \right) dt,$$

and can be obtained as

$$\theta = \mathrm{Leg}_L^* \theta,$$

where

$$\theta = p_i dq^i + p dt.$$

[1] We will deal with this annoying parameter later

Hamiltonian formalism in Classical Field Theory I

We have

$$t \sim x^\mu,\ q^i \sim \phi^i = y^i,\ \dot{q}^i \sim \frac{\partial \phi^i}{\partial x^\mu} = z^i_\mu.$$

The generalization of the Legendre transformation to fields is:

$$(x^\mu, y^i, z^i_\mu) \mapsto \left(z^\mu, y^i, p_i^\mu = \frac{\partial L}{\partial z^i_\mu}, p = -\frac{\partial L}{\partial z^i_\mu} z^i_\mu + L \right),$$

and the Poincaré-Cartan form

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x_\mu - \left(\frac{\partial L}{\partial z^i_\mu} z^i_\mu - L \right) d^n x$$

can be obtained as

$$\Theta_{\mathcal{L}} = \mathrm{Leg}^*_{\mathcal{L}}\, \Theta,$$

where

$$\Theta = p_i^\mu dy^i \wedge d^{n-1}x_\mu + p d^n x.$$

Can $\mathrm{Leg}_{\mathcal{L}}$ be interpreted as an intrinsic map?

Yes!

$$\mathrm{Leg}_{\mathcal{L}} : J^1\pi \to \bigwedge_2^n Y,$$

where

$$\bigwedge_2^n Y = \{\alpha \in \bigwedge^n Y,\ \iota_{e_1 \wedge e_2}\alpha = 0, e_i \in \ker d\pi\}.$$

Esentially, all forms that have the following local expression

$$\alpha = p d^n x + p_i^\mu dy^i \wedge d^{n-1}x_\mu.$$

Furthermore, we can define $\mathrm{Leg}_{\mathcal{L}}$ intrinsically (but won't).

Hamiltonian formalism in Classical Field Theory III

$$\Theta = p d^n x + p_i^\mu dy^i \wedge d^{n-1}x_\mu$$

is the tautological form on $\bigwedge_2^m Y$ intrinsically defined as

$$\Theta|_\alpha(v_1, \ldots, v_m) = \alpha(\tau_* v_1, \ldots, \tau_* v_m),$$

where $\alpha \in \bigwedge_2^m Y$, $v_1, \ldots, v_m \in T_\alpha \bigwedge_2^m Y$, and τ denotes the canonical projection $\tau : \bigwedge_2^m Y \to Y$.

With this formalism, the multisymplectic form is

$$\Omega = -dp \wedge d^n x - dp_i^\mu \wedge dy^i \wedge d^{n-1}x_\mu,$$

and satisfies

$$\Omega_{\mathcal{L}} = \mathrm{Leg}_{\mathcal{L}}^* \Omega.$$

There is a problem...

The Legendre transformation cannot define a diffeomorphism, since

$$\dim J^1\pi = \dim \bigwedge_2^m Y - 1.$$

However, quotienting

$$\bigwedge_2^m Y \xrightarrow{\tau} \bigwedge_2^m Y / \bigwedge_1^m Y,$$

where $\bigwedge_1^m Y$ is the set of forms with the expression $\alpha = p d^n x$, we can have

$$\dim J^1\pi = \dim \bigwedge_2^m Y / \bigwedge_1^m Y,$$

and we have the possibility for

$$\mathrm{leg}_{\mathcal{L}} := \tau \circ \mathrm{Leg}_{\mathcal{L}}$$

to be a diffeomorphism.

Hamiltonian formalism in Classical Field Theory V

In coordinates

$$\mathrm{leg}_{\mathcal{L}}^{*}\, p_i^{\mu} = \frac{\partial L}{\partial z_{\mu}^{i}},\ \mathrm{leg}_{\mathcal{L}}^{*}\, y^i = y^i,\ \mathrm{leg}_{\mathcal{L}}^{*}\, x^{\mu} = x^{\mu},$$

which in the case of Classical mechanics gives the original Legendre transformation (with no p involved).
When the Lagrangian is regular, that is, when $\mathrm{leg}_{\mathcal{L}}$ defines a diffeomorphism, we can define the Hamiltonian

$$h := \mathrm{Leg}_{\mathcal{L}} \circ \left(\mathrm{leg}_{\mathcal{L}}\right)^{-1},$$

which is a section of

$$\bigwedge_2^m Y \xrightarrow{\tau} \bigwedge_2^m Y / \bigwedge_1^m Y.$$

Locally,

$$h(x^{\mu}, y^i, p_i^{\mu}) = \left(x^{\mu}, y^i, p_i^{\mu}, p = -H = -\frac{\partial L}{\partial z_{\mu}^{i}} z_{\mu}^{i} + L\right)$$

Hamiltonian formalism in Classical Field Theory VI

Theorem
In the regular case, solving the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial L}{\partial z^i_\mu}\right) = \frac{\partial L}{\partial y^i}, \text{ or } \; (j^1\phi)^*\iota_\xi\Omega_{\mathcal{L}} = 0, \forall \xi \in \mathfrak{X}(J^1\pi)$$

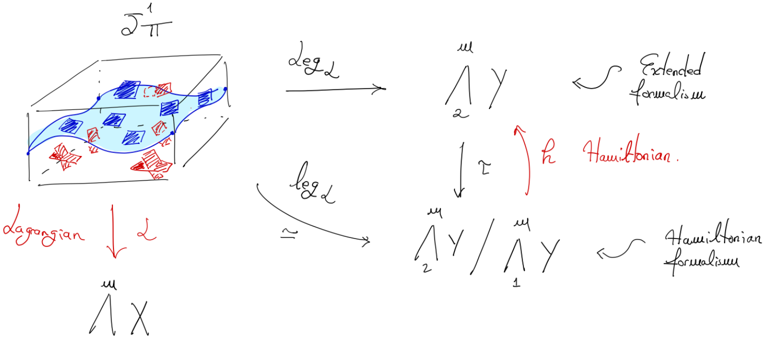
is equivalent to solving the Hamilton-De Donder-Weyl equations

$$\frac{\partial y^i}{\partial x^\mu} = \frac{\partial H}{\partial p^\mu_i}, \; \frac{\partial p^\mu_i}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}, \text{ or } \psi^*\iota_\xi\Omega_h = 0, \; \forall \xi \in \mathfrak{X}\left(\bigwedge_2^m Y / \bigwedge_1^m Y\right),$$

for $\psi : X \to \bigwedge_2^m Y / \bigwedge_1^m Y$ a section. The correspondence is given by

$$\psi = \mathrm{leg}_{\mathcal{L}} \circ j^1\phi.$$

Summary (regular case)



The following forms characterize the original variational problem

$$\Omega_h = h^*\Omega,$$

$$\Omega_{\mathcal{L}} = \operatorname{Leg}_{\mathcal{L}}^* \Omega = \operatorname{leg}_{\mathcal{L}}^* \Omega_h.$$

Multisymplectic Geometry and graded Poisson Geometry

Definition
A multisymplectic manifold of order n is a pair (M, ω), where M is a smooth manifold, and ω is a closed $(n+1)-$form.

An immediate example is the bundle of n-forms on a manifold Q.

$$M := \bigwedge^n T^* Q \xrightarrow{\tau} Q$$

has a canonical n-form,

$$\Theta|_\alpha(v_1, \ldots, v_n) := \alpha(\tau_* v_1, \ldots, \tau_* v_n)$$

and

$$\Omega := -d\Theta$$

defines a multisymplectic structure on M.

Basic definitions II

Definition
Let (M, ω) be a multisymplectic manifold of order n. A q-multivector field U on M ($q \leq n$) is called Hamiltonian if

$$\iota\omega = d\alpha,$$

for certain $(n-q)$-form α, which will also be called Hamiltonian.

Denote by

$$\Omega_H^{a-1}(M)$$

the space of all Hamiltonian forms.

- Top degree Hamiltonian multivector fields (n-multivector fields) represent solutions to the variational problem,

$$\iota_U\omega = dH, H \in C^\infty(M).$$

- Hamiltonian vector fields $X \in \mathfrak{X}(M)$ are symmetries, $\pounds_X\omega = 0$ and the corresponding $(n-1)-$form can be thought of as the Noether current of the symmetry.

Brackets I

The Poisson bracket from Classical Mechanics is:

$$\{f, g\}^{\bullet} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial q^i},$$

which is characterized by the Poisson bivector

$$\Lambda = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}, \{f, g\}^{\bullet} = \Lambda(df, dg).$$

Definition (Poisson bracket)
A Poisson bracket on M is a Lie algebra structure on $C^{\infty}(M)$, $\{\cdot, \cdot\}^{\bullet}$ that also satisfies Leibniz identity $\{fg, h\}^{\bullet} = f\{g, h\}^{\bullet} + g\{f, h\}^{\bullet}$.

In fact, there is an equivalence

$$\{\text{Poisson brackets}\} \cong \{\text{Bivector fields } \Lambda \text{ satisfying } [\Lambda, \Lambda] = 0\}$$

Brackets II

We want to find the analogue concepts in multisymplectic field theory:

1. Find the concept of brackets in field theory;
2. Find the tensorial anaolgue to the Poisson bivector;
3. Prove these two coincide.

Brackets III

Proposition
Let (M,ω) be a multisymplectic manifold and α, β be Hamiltonian forms, with Hamiltonian multivector fields, X, Y, respectively. Then

$$\{\alpha,\beta\} := (-1)^{k-1-\operatorname{ord}\beta}\iota_Y\iota_X\omega$$

is a Hamiltonian form. Its Hamiltonian multivector field is $-[X,Y]$ (the Schouten-Nijenhuis bracket).

Definition
Define the Poisson bracket of two Hamiltonian forms by

$$\{\alpha,\beta\} := (-1)^{(k-1-\operatorname{ord}\beta)}\iota_Y\iota_X\omega,$$

which is again Hamiltonian by the previous proposition.

Brackets IV

Defining a new notion of degree, $\deg\alpha := k - 1 - \operatorname{ord}\alpha$, the bracket satisfies:

- *It is graded:*

$$\deg\{\alpha,\beta\} = \deg\alpha + \deg\beta;$$

- *It is graded-skew-symmetric:*

$$\{\alpha,\beta\} = -(-1)^{\deg\alpha\deg\beta}\{\beta,\alpha\};$$

- *It is local:* If $d\alpha|_x = 0$, $\{\alpha,\beta\}|_x = 0$

Brackets V

- *It satisfies Leibniz identity:* Let $\beta^j \in \Omega_H^{b-1}(M), \gamma_j \in \Omega_H^{c-1}(M)$. If $\beta^j \wedge d\gamma \in \Omega_H^{b+c-1}(M)$, then, for $a = k$,

$$\{\beta^j \wedge d\gamma_j, \alpha\} = \{\beta^j, \alpha\} \wedge d\gamma_j + (-1)^{k-\deg \beta^j} d\beta \wedge \{\gamma_j, \alpha\};$$

- *It is invariant by symmetries:* If $X \in \mathfrak{X}(M)$ and $\pounds_X \alpha = 0$, then $\iota_X \alpha \in \Omega_H^{a-2}(M)$ and

$$\{\iota_X \alpha, \beta\} = (-1)^{\deg \beta} \iota_X \{\alpha, \beta^j\};$$

- *It satisfies graded Jacobi identity (up to an exact term):*

$$(-1)^{\deg \alpha \deg \gamma} \{\{\alpha, \beta\}, \gamma\} + \text{cyclic terms} = \text{exact form}.$$

Brackets VI

Then we can define:

Definition (Poisson bracket)
Let $S^k \subseteq \bigwedge^k M$ be a nondegenerate[2] subbundle satisfying that

$$S^a := \{\alpha = \iota_U \gamma, U \in \bigvee_{k-a} M, \alpha \in S^k\} \subseteq \bigwedge^a M$$

defines a subbundle for each a. Then, a Poisson bracket is a bilinear operation

$$\Omega_H^{a-1}(M) \otimes \Omega_H^{b-1}(M) \xrightarrow{\{\cdot,\cdot\}} \Omega^{a+b-(k+1)}(M)$$

that satisfies the previous list of properties.

[2]that is, $\iota_v S^k = 0, v \in TM$ implies $v = 0$

What is the analogue to the Poisson tensor?

In Symplectic Geometry, the Poisson bivector can be thought of as the inverse of the map induced by the contraction

$$TM \to T^*M, v \mapsto \iota_v\omega.$$

Then, the natural analogue would be the "inverse" of

$$TM \to \bigwedge^k M,\ v \mapsto \iota_v\omega.$$

Definition (Almost Poisson tensor)
An almost Poisson tensor of order k is am skew-symmetric linear bundle map

$$\sharp : S^k \to TM,$$

where $S^k \subseteq \bigwedge^k M$ is a non-degenerate subbundle.

What about integrability?

Definition (Poisson tensor)
We say that an almost Poisson tensor of order k

$$\sharp : S^k \to TM$$

is integrable if it satisfies the following property. For α, β taking values in S^k, if we define

$$\theta := \pounds_{\sharp(\alpha)}\beta - \iota_{\sharp(\beta)}d\alpha,$$

we have $\theta \in S^k$ and

$$\sharp(\theta) = [\sharp(\alpha), \sharp(\beta)].$$

In this case we call $\sharp : S^k \to TM$ a Poisson tensor.

Brackets IX

Then we have the following results:

Theorem
A Poisson tensor $\sharp : S^k \to M$ *determines a multisymplectic foliation of* M, $(\mathcal{F}, \omega_{\mathcal{F}})$.

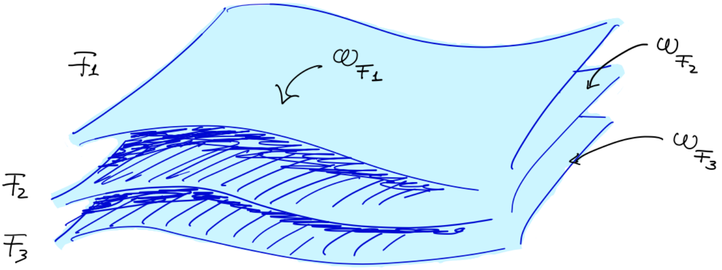
Theorem
Given a non-degenrate multisymplectic foliation $(F, \omega_{\mathcal{F}})$, *if*

$$\dim_x \mathcal{F} - \binom{\dim_x \mathcal{F}}{k}$$

remains constant on M, *it arises from a (non-necessarily unique) Poisson tensor.*

Brackets X

Given a Poisson tensor $\sharp : S^k \to TM$, we have:



$$d\omega_{\mathcal{F}_i} = 0$$

Brackets XI

Furthermore,

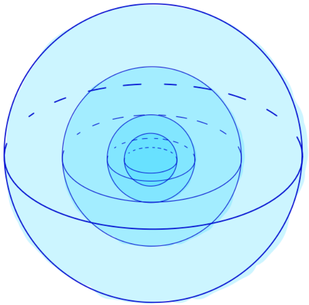
Theorem
Under certain integrability conditions on the space S^k (which applies, for instance, in the Extended Hamiltonian formalism), we have

$$\{\textit{Poisson brackets of order } k\} \cong \{\textit{Poisson tensors of order } k\}$$

As an academic example, a manifold M foliated by $(k+1)$-dimensional manifolds together with volume forms admit a Poisson tensor of order k. And, in particular, have associated the Poisson bracket as an algebraic invariant.

Brackets XII

As an example:



$r\,\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$

con la forma de volumen:

$$\omega\big|_x = c_x \, dx^1 \wedge \dots \wedge dx^n$$

Conclusions

Conclusions

- The multisymplectic formalism of Classical Field Theories gives an intrinsic formulation of the variational problem.
- It is completely characterized by a closed $(n+1)-$form, $\Omega_{\mathcal{L}}$, called the multisymplectic form.
- This motivates an abstract study of manifold together with closed forms (M, ω).
- Although too general, this study allows for a better understanding of Classical Field Theory (and some geometric byproducts are obtained).
- A recent example can be seen in the generalization of the geometry of Poisson brackets to multisymplectic manifolds, giving the correct notion of both the Poisson bracket and tensor.

Thank you for your attention!