A graded aproach to brackets in classical field theory

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Structure of the talk

Poisson an Dirac structures in Classical Mechanics

Summary of multisymplectic field theory

Graded Poisson and Dirac structures

Poisson an Dirac structures in Classical Mechanics

Poisson structures

Definition
A Poisson structure on a smooth manifold M is a bracket $\{\cdot,\cdot\}$ defined on $C^\infty(M)$ that satisfies:

- Skew-symmetry, $\{f,g\} = -\{g,f\}$
- Leibniz identity, $\{fh,g\} = f\{h,g\} + h\{f,g\}$
- Jacobi identity, $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0.$

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Going degenerate: Dirac structures

Definition
A Dirac structure on a smooth manifold M is a bracket $\{\cdot,\cdot\}$ defined on

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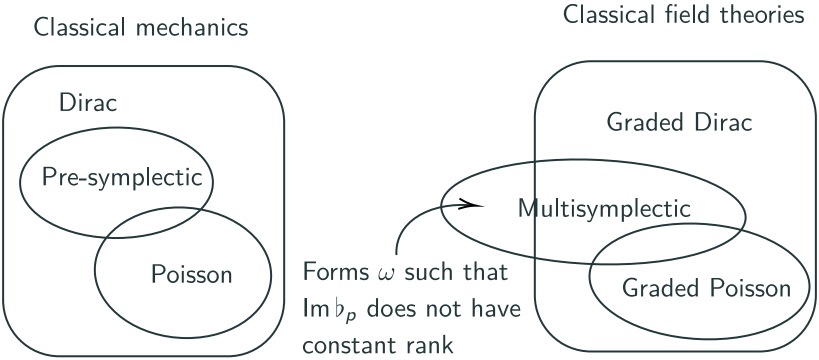
- A foliation of M by pre-symplectic leaves.

Objective

Translate this picture to classical field theories.

More particularly,

Find an equivalence between brackets and some tensorial object.



Summary of multisymplectic field theory

The geometric setting

- Fields, denoted by ϕ, are sections of a fibered manifold $Y \xrightarrow{\pi} X$.

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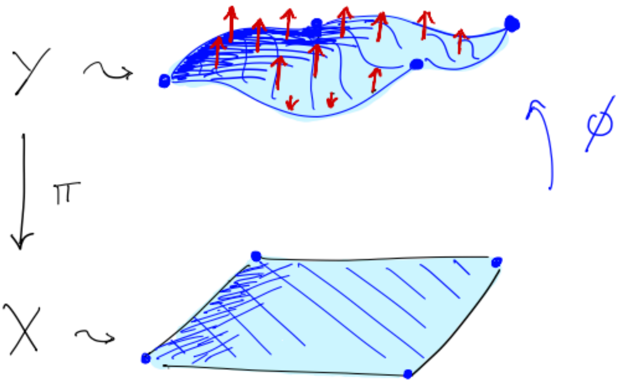
- Fields, denoted by ϕ, are sections of a fibered manifold $Y \xrightarrow{\pi} X$.
- A first order variational problem is defined through a Lagrangian density $\mathcal{L}$ on $J^1\pi_{YX}$ (which defines an n-form on X at each point), with local expression

$$\mathcal{L} = L(x^\mu, y^i, z^i_\mu)d^n x,$$

and the action can be expressed as

$$\mathcal{J}[\phi] = \int_X (j^1\phi)^* \mathcal{L}.$$

$$\xi := \frac{d}{dt}\Big|_{t=0} \phi_t$$



Stationary sections

- We define the multisymplectic form as

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}},$$

where

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x_\mu - \left(z^i_\mu \frac{\partial L}{\partial z^i_\mu} - L \right) d^n x$$

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  is the Poincaré-Cartan form.
- Stationary fields (solutions to the field equations) are characterized by

$$(j^1\phi)^* \iota_\eta \Omega_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi).$$

In coordinates,

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu} \left(\frac{\partial L}{\partial z^i_\mu} \right) = \frac{\partial L}{\partial y^i}.$$

Hamiltonian formalism I

- The extended Hamiltonian formalism takes place in

$$\bigwedge_2^n Y = \{\alpha \in \bigwedge^n T^*Y : \iota_{e_1 \wedge e_2}\alpha = 0, \text{ where } e_i \in \ker d\pi\},$$

$\pi : Y \to X$. Locally, these forms can be expressed as

$$\alpha = p d^n x + p_i^\mu dy^i \wedge d^{n-1}x_\mu.$$

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$$\alpha = p d^n x + p_i^\mu dy^i \wedge d^{n-1}x_\mu.$$

- Therefore, we have coordinates (x^μ, y^i, p_i^μ, p) representing the previous form. We have a canonical multisymplectic structure,

$$\Omega = -dp \wedge d^n x - dp_i^\mu \wedge dy^i \wedge d^{n-1}x_\mu.$$

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- We can obtain the Poincaré cartan form on $J^1\pi$ as

$$\Omega_{\mathcal{L}} = \mathrm{Leg}_{\mathcal{L}}^* \Omega,$$

  where $\mathrm{Leg}_{\mathcal{L}}(x^\mu, y^i, z_\mu^i) = (x^\mu, y^i, \frac{\partial L}{\partial z_\mu^i}, L - z_\mu^i \frac{\partial L}{\partial z_\mu^i})$ is the Legendre transformation.

Hamiltonian formalism II

- We also have a reduced Hamiltonian formalism, which takes place in

$$Z^* = \bigwedge_2^n Y / \bigwedge_1^n Y,$$

where $\{\alpha \in \bigwedge^n T^*M : \iota_e \alpha = 0, e \in \ker d\pi\}$, locally, $\alpha = p d^n x$. So we have natural coordinates (x, y^i, p_i^μ).

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- Then, we obtain the reduced Legendre transformation $\mathrm{leg}_{\mathcal{L}} := \tau \circ \mathrm{Leg}_{\mathcal{L}}$, where

$$\tau : \bigwedge_2^n Y \to Z^*$$

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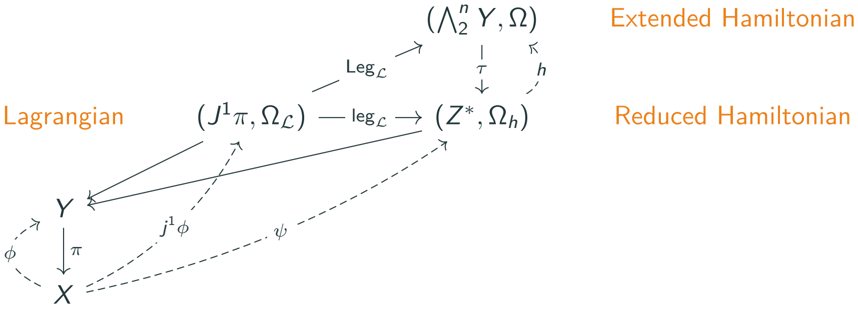
- In order to obtain the field theory on Z^* we need a Hamiltonian section

$$h : Z^* \to \bigwedge_2^n Y.$$

Equivalence (with regular Lagrangians)



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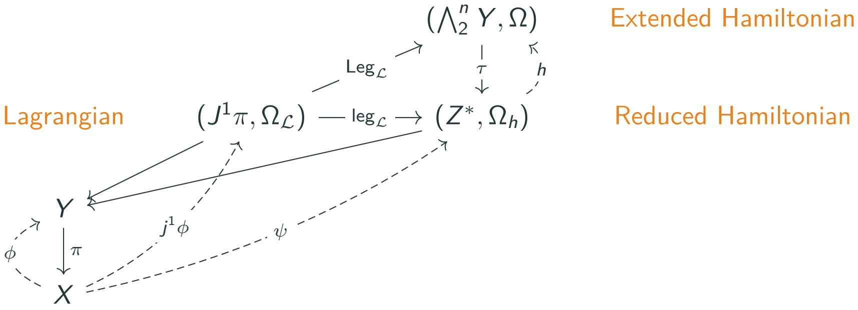

$$\phi \text{ is stationary} \Leftrightarrow (j^1\phi)^* \iota_\xi \Omega_{\mathcal{L}} = 0 \Leftrightarrow \psi^* \iota_\xi \Omega_h = 0.$$

In coordinates,

$$\frac{\partial \psi_i^\mu}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}, \; \frac{\partial \psi^i}{\partial x^\mu} = \frac{\partial H}{\partial p_\mu^i}.$$

Graded Poisson and Dirac structures

Multisymplectic manifolds

In order to work with any of the formalisms (Lagrangian, extended Hamiltonian, reduced Hamiltonian), we will work abstractly with a multisymplectic manifold, (M, ω), a manifold M together with a closed $(n+1)-$form.

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In order to work with any of the formalisms (Lagrangian, extended Hamiltonian, reduced Hamiltonian), we will work abstractly with a multisymplectic manifold, (M, ω), a manifold M together with a closed $(n+1)-$form.

We have already a lot of examples:

- $(J^1\pi, \Omega_{\mathcal{L}})$, where $\mathcal{L}$ is a Lagrangian;
- $(\bigwedge_2^n Y, \Omega)$;
- (Z^*, Ω_h), where $h : Z^* \to \bigwedge_2^n Y$ is a Hamiltonian section;
- (M, ω), where M is an orientable manifold and ω is a volume form.

Hamiltonian multivector fields and Hamiltonian forms

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We have the following:

Proposition
If U, V are Hamiltonian multivector fields of degree p, q. Then $[U, V]$ is a Hamiltonian multivector field of order $p + q - 1$. The corresponding Hamiltonian form is

$$(-1)^q \iota_{U \wedge V} \Omega.$$

Graded Poisson brackets

The previous proposition induces the following:

Definition
Let α, β be Hamiltonian forms of order $k - p, k - q$, respectively. Define their Poisson bracket

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Then, we have

Theorem
Modulo exact forms, the previous brackets defines a graded Lie algebra on the space of Hamiltonian forms

Question: Does this recover the multisymplectic form?

Properties of graded Poisson brackets

If we set $\deg \beta := k -$ order of β, then th Poisson bracket satisfies:

- *It is graded:*

$$\deg\{\alpha, \beta\} = \deg \alpha + \deg \beta;$$

- *It is graded-skew-symmetric:*

$$\{\alpha, \beta\} = -(-1)^{\deg \alpha \deg \beta}\{\beta, \alpha\};$$

- *It is local:* If $d\alpha|_x = 0$, $\{\alpha, \beta\}|_x = 0$
- *It satisfies graded Jacobi identity (up to an exact term):*

$$(-1)^{\deg \alpha \deg \gamma}\{\{\alpha, \beta\}, \gamma\} + \text{cyclic terms} = \text{exact form}.$$

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- *It satisfies Leibniz identity:* For $a = k$, if $\beta \wedge d\gamma \in \Omega_H^{b+c-1}(M)$, then

$$\{\beta \wedge d\gamma, \alpha\} = \{\beta, \alpha\} \wedge d\gamma + (-1)^{k-\deg \beta} d\beta \wedge \{\gamma, \alpha\};$$

- *It is invariant by symmetries:* If $X \in \mathfrak{X}(M)$ and $\pounds_X \alpha = 0$, then $\iota_X \alpha \in \Omega_H^{a-2}(M)$ and

$$\{\iota_X \alpha, \beta\} = (-1)^{\deg \beta} \iota_X\{\alpha, \beta\};$$

Onto the definitions...

First, we look at the linearized version:

Definition
Let M be a manifold. A graded Poisson structure of order n is a tuple $(S^a, K_p, \sharp_a)$, where $S^a \subseteq \bigwedge^a M$ is a vector subbundle of forms, $K_p \subseteq \bigvee_p M (= \bigwedge^p TM)$ is a subbundle of multivectors, and

$$\sharp_a : S^a \to \bigvee_{n+1-a} M/K_{n+1-a}$$

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are linear bundle maps sastifying:

- $K_p = (S^a)^{\circ, p}$, for $p \leq a$ and $K_1 = 0$.
- The maps $\sharp_a$ are *skew-symmetric*, that is,

  $$\iota_{\sharp_a(\alpha)} \beta = (-1)^{(n+1-a)(n+1-b)} \iota_{\sharp_b(\beta)} \alpha,$$

  for all $\alpha \in S^a$, $\beta \in S^b$.

And it is integrable:

- It is *integrable:* For $\alpha : M \to S^a$, $\beta : M \to S^b$ sections such that $a + b \leq 2n + 1$, and U, V multivectors of order $p = n + 1 - a$, $q = n + 1 - b$, respectively such that

$$\sharp_a(\alpha) = U + K_p, \ \sharp_b(\beta) = V + K_q,$$

we have that the $(a + b - k)$-form

$$\theta := (-1)^{(p-1)q} \pounds_U \beta + (-1)^q \pounds_V \alpha - \frac{(-1)^q}{2} d \left(\iota_V \alpha + (-1)^{pq} \iota_U \beta \right)$$

takes values in S_{a+b-k}, and

$$\sharp_{a+b-k}(\theta) = [U, V] + K_{p+q-1}.$$

Field theories as an example

Let (M, ω) be a non-degenerate multisymplectic manifold of order n, that is, the form ω must satisfy $\iota_v \omega = 0$ if and only if $v = 0$. Then, the (linear) correspondence between Hamiltonian multivector fields and forms defines a graded Poisson structure of order n:

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- $$S^a = \{\iota_U \omega : U \in \bigvee_{n+1-a} M\};$$
- $$K_p = \ker_p \omega;$$
- $\sharp_a : S^a \to \bigvee_{n+1-a} / K_{n+1-a}$ is given by

$$\sharp_a(\alpha) = U + K_{n+1-a} \text{ if and only if } \iota_U \omega = \alpha.$$

In this case, $\sharp_a$ are the inverse of the $\flat_p$ (contraction) maps induced by ω.

Relationship with graded Poisson brackets

Given a graded Poisson structure on M, $(S^a, \sharp_a, K_p)$, we can define a Hamiltonian form as an $(a-1)-$form, α such that $d\alpha \in S^a$.

Definition
The Poisson bracket of Hamiltonian forms is given by

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It satisfies all previous properties and, furthermore,

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Theorem
Under some integrability conditions on the sequence of subspaces S^a, any graded Poisson structure on this family is completely characterized by the graded Poisson bracket it induces. That is, we get a 1-1 correspondence

$$\{\textit{Graded Poisson structures}\} \cong \{\textit{Graded Poisson brackets}\}.$$

What about Dirac strcutures?

- We should think about graded Dirac structures as a generalization of both graded Poisson and multisymplectic;
- Non-degeneracy in graded Poisson structures is given by $K_1 = 0$, or rather,

$$\sharp_k : S^k \to TM.$$

- In order to obtain graded Dirac, we just allow for non-trivial K_1, and hence we have the "same" definition

$$\sharp_a : S^a \to \bigvee_{n+1-a} M/K_{n+1-a};$$

- Turns out that the non-degeneracy is non-essential.

Brackets in graded Dirac structures

Theorem
Defining

$$\{\alpha, \beta\} := (-1)^{\deg \beta} \iota_{\sharp_a d\alpha} d\beta,$$

we get a graded Poisson bracket on the space of Hamiltonian forms (defined in the same way).

Furthermore:

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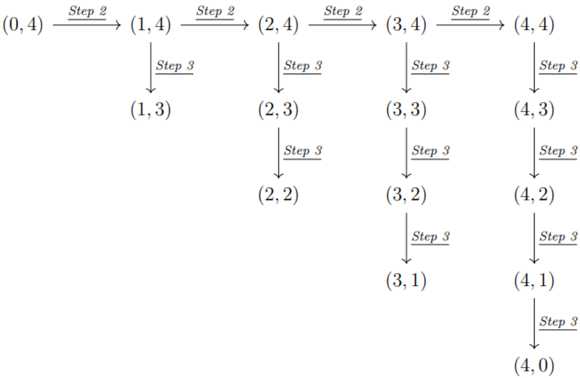
Remark
In this sense, these definitions respect the flavour of Poisson and Dirac geometry, giving a linearized version of brackets. In the first case, a bracket defined in a non-degenrate space of forms; and allowing for degeneracy in the second.

Integrability conditions

What are the integrability conditions on the sequence S^a?

- Locally, there exists Hamiltonian forms $\gamma_{ij} \in \Omega_H^{b-2}(U)$, and functions f_i^j such that
$$S^b = \langle df_i^j \wedge d\gamma_{ij}, i \rangle;$$
- For each $1 \leq a \leq k$, locally, there exists a family of Hamiltonian forms forms γ^j, and a family of vector fields X^j such that
$$S^a = \langle d\gamma^j \rangle \; \pounds_{X^j} \gamma^j = 0,$$
and
$$S^{a-1} = \langle d\iota_{X^j} \gamma^j \rangle.$$

Idea of the Proof



Final remarks and future research

- We developed the theory of Poisson bracket and tensors in classical field theories;
- Can we find an analogue to Lie-Poisson structures in this setting?;
- How are these bracket and structures related to reduction?

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Thank you for your attention!