Graded Poisson brackets in classical field theories

Q-Math Seminar

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ICMAT-UNIR

Structure of the talk

Poisson and Dirac structures in Classical Mechanics

Summary of multisymplectic field theory

Graded Poisson and Dirac structures

Dynamics: Application to almost-regular Lagrangians (work in progress)

Poisson and Dirac structures in Classical Mechanics

Poisson structures

Definition
A Poisson structure on a smooth manifold M is a bracket $\{\cdot,\cdot\}$ defined on $C^\infty(M)$ that satisfies:

- Skew-symmetry, $\{f,g\} = -\{g,f\}$
- Leibniz identity, $\{fh,g\} = f\{h,g\} + h\{f,g\}$
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Going degenerate: Dirac structures

Definition

A Dirac structure on a smooth manifold M is a bracket $\{\cdot,\cdot\}$ defined on

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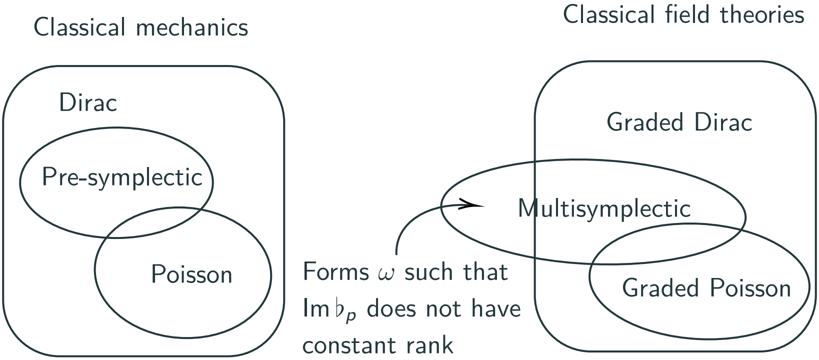
- A foliation of M by pre-symplectic leaves.

Objective

Translate this picture to classical field theories.

More particularly,

Find an equivalence between brackets and some tensorial object.



Summary of multisymplectic field theory

The geometric setting

- Fields, denoted by ϕ, are sections of a fibered manifold $Y \xrightarrow{\pi} X$.

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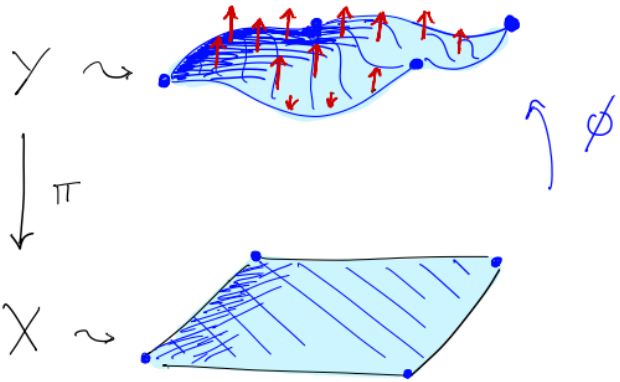
- A first order variational problem is defined through a Lagrangian density $\mathcal{L}$ on $J^1\pi_{YX}$ (which defines an n-form on X at each point), with local expression

$$\mathcal{L} = L(x^\mu, y^i, z^i_\mu) d^n x,$$

and the action can be expressed as

$$\mathcal{J}[\phi] = \int_X (j^1\phi)^* \mathcal{L}.$$

$$\xi := \frac{d}{dt}\Big|_{t=0} \phi_t$$



Stationary sections

- We define the multisymplectic form as

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}},$$

where

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x_\mu - \left(z^i_\mu \frac{\partial L}{\partial z^i_\mu} - L \right) d^n x$$

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- Stationary fields (solutions to the field equations) are characterized by

$$(j^1\phi)^* \iota_\eta \Omega_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi).$$

In coordinates,

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial L}{\partial z^i_\mu}\right) = \frac{\partial L}{\partial y^i}.$$

Hamiltonian formalism I

- The extended Hamiltonian formalism takes place in

$$\bigwedge_2^n Y = \{\alpha \in \bigwedge^n T^*Y : \iota_{e_1 \wedge e_2}\alpha = 0, \text{ where } e_i \in \ker d\pi\},$$

$\pi : Y \to X$. Locally, these forms can be expressed as

$$\alpha = p d^n x + p_i^\mu dy^i \wedge d^{n-1} x_\mu.$$

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$$\alpha = pd^n x + p_i^\mu dy^i \wedge d^{n-1}x_\mu.$$

- Therefore, we have coordinates (x^μ, y^i, p_i^μ, p) representing the previous form. We have a canonical multisymplectic structure,

$$\Omega = -dp \wedge d^n x - dp_i^\mu \wedge dy^i \wedge d^{n-1}x_\mu.$$

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$$\Omega = -dp \wedge d^n x - dp_i^\mu \wedge dy^i \wedge d^{n-1}x_\mu.$$

- We can obtain the Poincaré cartan form on $J^1\pi$ as

$$\Omega_{\mathcal{L}} = \mathrm{Leg}_{\mathcal{L}}^* \Omega,$$

where $\mathrm{Leg}_{\mathcal{L}}(x^\mu, y^i, z_\mu^i) = (x^\mu, y^i, \frac{\partial L}{\partial z_\mu^i}, L - z_\mu^i \frac{\partial L}{\partial z_\mu^i})$ is the Legendre transformation.

Hamiltonian formalism II

- We also have a reduced Hamiltonian formalism, which takes place in

$$Z^* = \bigwedge_2^n Y / \bigwedge_1^n Y,$$

where $\{\alpha \in \bigwedge^n T^*M : \iota_e \alpha = 0, e \in \ker d\pi\}$, locally, $\alpha = p d^n x$. So we have natural coordinates (x, y^i, p_i^μ).

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- Then, we obtain the reduced Legendre transformation $\mathrm{leg}_{\mathcal{L}} := \tau \circ \mathrm{Leg}_{\mathcal{L}}$, where

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denotes the projection.

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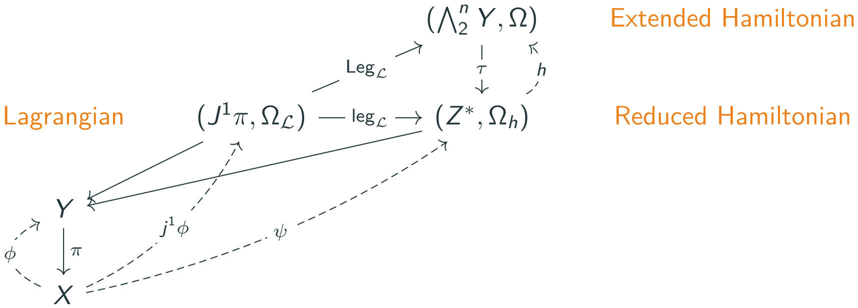
- In order to obtain the field theory on Z^* we need a Hamiltonian section

$$h : Z^* \to \bigwedge_2^n Y.$$

Equivalence (with regular Lagrangians)



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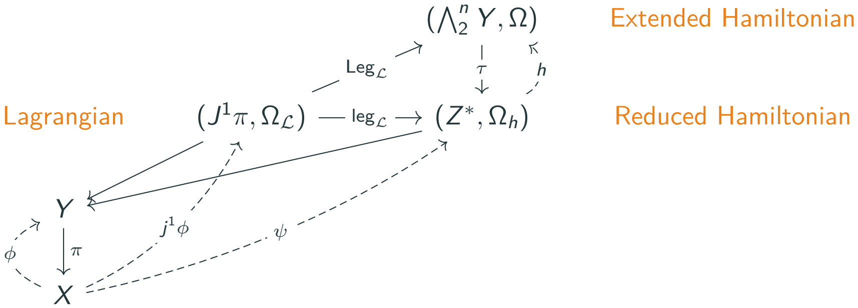

$$\phi \text{ is stationary} \Leftrightarrow (j^1\phi)^*\iota_\xi\Omega_{\mathcal{L}} = 0 \Leftrightarrow \psi^*\iota_\xi\Omega_h = 0.$$

In coordinates,

$$\frac{\partial \psi_i^\mu}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}, \; \frac{\partial \psi^i}{\partial x^\mu} = \frac{\partial H}{\partial p_\mu^i}.$$

Graded Poisson and Dirac structures

Multisymplectic manifolds

In order to work with any of the formalisms (Lagrangian, extended Hamiltonian, reduced Hamiltonian), we will work abstractly with a multisymplectic manifold, (M, ω), a manifold M together with a closed $(n+1)-$form.

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We have already a lot of examples:

- $(J^1\pi, \Omega_{\mathcal{L}})$, where $\mathcal{L}$ is a Lagrangian;
- $(\bigwedge_2^n Y, \Omega)$;
- (Z^*, Ω_h), where $h : Z^* \to \bigwedge_2^n Y$ is a Hamiltonian section;
- (M, ω), where M is an orientable manifold and ω is a volume form.

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We have the following:

Proposition
If U, V are Hamiltonian multivector fields of degree p, q. Then $[U, V]$ is a Hamiltonian multivector field of order $p + q - 1$. The corresponding Hamiltonian form is

$$(-1)^q \iota_{U \wedge V} \Omega.$$

Graded Poisson brackets

The previous proposition induces the following:

Definition
Let α, β be Hamiltonian forms of order $k-p, k-q$, respectively. Define their Poisson bracket

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Then, we have

Theorem
Modulo exact forms, the previous brackets defines a graded Lie algebra on the space of Hamiltonian forms

Question: Does this recover the multisymplectic form?

Properties of graded Poisson brackets

If we set $\deg \beta := k -$ order of β, then th Poisson bracket satisfies:

- *It is graded:*

$$\deg\{\alpha, \beta\} = \deg \alpha + \deg \beta;$$

- *It is graded-skew-symmetric:*

$$\{\alpha, \beta\} = -(-1)^{\deg \alpha \deg \beta}\{\beta, \alpha\};$$

- *It is local:* If $d\alpha|_x = 0$, $\{\alpha, \beta\}|_x = 0$
- *It satisfies graded Jacobi identity (up to an exact term):*

$$(-1)^{\deg \alpha \deg \gamma}\{\{\alpha, \beta\}, \gamma\} + \text{cyclic terms} = \text{exact form}.$$

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- *It satisfies Leibniz identity:* For $a = k$, if $\beta \wedge d\gamma \in \Omega_H^{b+c-1}(M)$, then

$$\{\beta \wedge d\gamma, \alpha\} = \{\beta, \alpha\} \wedge d\gamma + (-1)^{k-\deg \beta} d\beta \wedge \{\gamma, \alpha\};$$

- *It is invariant by symmetries:* If $X \in \mathfrak{X}(M)$ and $\pounds_X \alpha = 0$, then $\iota_X \alpha \in \Omega_H^{a-2}(M)$ and

$$\{\iota_X \alpha, \beta\} = (-1)^{\deg \beta} \iota_X \{\alpha, \beta\};$$

Onto the definitions...

First, we look at the linearized version:

Definition
Let M be a manifold. A graded Dirac structure of order n is a tuple $(S^a, K_p, \sharp_a)$, where $S^a \subseteq \bigwedge^a M$ is a vector subbundle of forms, $K_p \subseteq \bigvee_p M (= \bigwedge^p TM)$ is a subbundle of multivectors, and

$$\sharp_a : S^a \to \bigvee_{n+1-a} M / K_{n+1-a}$$

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are linear bundle maps sastifying:

- $K_p = (S^a)^{\circ, p}$, for $p \leq a$.
- The maps $\sharp_a$ are *skew-symmetric*, that is,

  $$\iota_{\sharp_a(\alpha)}\beta = (-1)^{(n+1-a)(n+1-b)}\iota_{\sharp_b(\beta)}\alpha,$$

  for all $\alpha \in S^a$, $\beta \in S^b$.

And it is integrable:

- It is *integrable:* For $\alpha : M \to S^a$, $\beta : M \to S^b$ sections such that $a + b \leq 2n + 1$, and U, V multivectors of order $p = n + 1 - a$, $q = n + 1 - b$, respectively such that

$$\sharp_a(\alpha) = U + K_p, \ \sharp_b(\beta) = V + K_q,$$

we have that the $(a + b - k)$-form

$$\theta := (-1)^{(p-1)q} \pounds_U \beta + (-1)^q \pounds_V \alpha - \frac{(-1)^q}{2} d \left(\iota_V \alpha + (-1)^{pq} \iota_U \beta \right)$$

takes values in S_{a+b-k}, and

$$\sharp_{a+b-k}(\theta) = [U, V] + K_{p+q-1}.$$

Field theories as an example

Let (M, ω) be a multisymplectic manifold of order n, that is, the form ω must satisfy $\iota_v \omega = 0$ if and only if $v = 0$. Then, the (linear) correspondence between Hamiltonian multivector fields and forms defines a graded Dirac structure of order n:

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- $$S^a = \{\iota_U \omega : U \in \bigvee_{n+1-a} M\};$$
- $$K_p = \ker_p \omega;$$
- $\sharp_a : S^a \to \bigvee_{n+1-a} / K_{n+1-a}$ is given by
  $$\sharp_a(\alpha) = U + K_{n+1-a} \text{ if and only if } \iota_U \omega = \alpha.$$

In this case, $\sharp_a$ are the inverse of the $\flat_p$ (contraction) maps induced by ω.

Relationship with graded Poisson brackets

Given a graded Dirac structure on M, $(S^a, \sharp_a, K_p)$, we can define a Hamiltonian form as an $(a-1)-$form, α such that $d\alpha \in S^a$.

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The Poisson bracket of Hamiltonian forms is given by

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Theorem
Under some integrability conditions on the sequence of subspaces S^a, any graded Dirac structure on this family is completely characterized by the graded Poisson bracket it induces. That is, we get a 1-1 correspondence

$$\{\textit{Graded Poisson structures}\} \cong \{\textit{Graded Poisson brackets}\}.$$

But why?

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- Tools.

	Symplectic	Poisson	Dirac
Easily restricted?	Yes	No	Yes
Easily Quotiented?	No	Yes	Yes

	Multisymplectic	Graded Poisson	Graded Dirac
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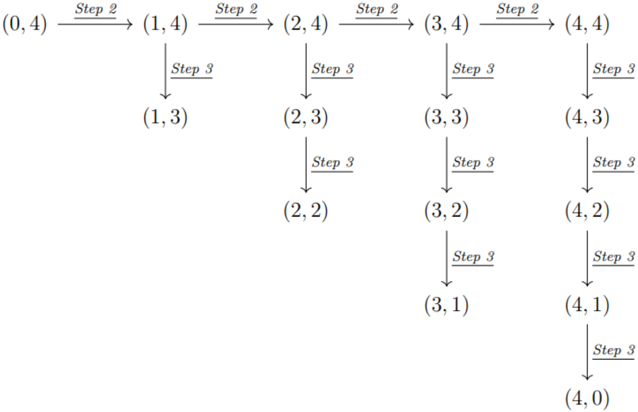
- For each $1 \leq a \leq k$, locally, there exists a family of Hamiltonian forms forms γ^j, and a family of vector fields X^j such that

$$S^a = \langle d\gamma^j \rangle \ \pounds_{X^j}\gamma^j = 0,$$

  and

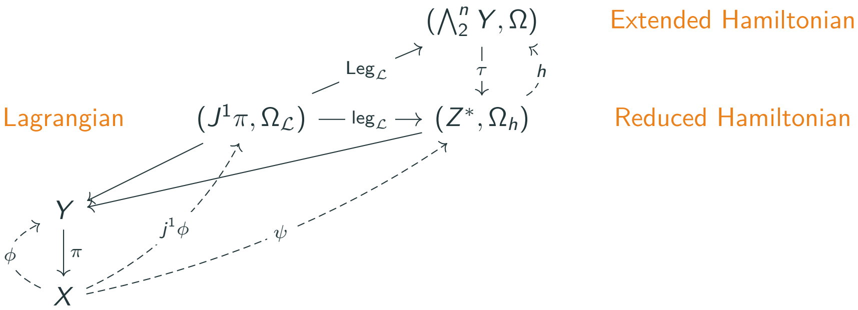
$$S^{a-1} = \langle d\iota_{X^j}\gamma^j \rangle.$$

Idea of the Proof



Dynamics: Application to almost-regular Lagrangians (work in progress)

Recall:



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- BF-theory: Given a principal bundle $\mathbb{P}\to M^{(4)}$,

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If we aim to do field theory in the Hamiltonian setting, we need to incorporate singular Lagrangians.

Proposed framework:

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- Calculate its image $\text{leg}_{\mathcal{L}}(J^1\pi) \subseteq \bigwedge_2^n Y / \bigwedge_1^n Y$.
- Restrict the graded Poisson structure to a graded Dirac structure on $\text{leg}_{\mathcal{L}}(J^1\pi)$.

Dynamics on graded Dirac manifolds I

Definition (Fibered graded Dirac manifold)
Let $\tau : M \to X$ be a fibered manifold. A fibered graded Dirac structure on M is a graded Dirac structure on M, $\sharp_n : S^n \to TM/K_1$ such that:

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Define $S^{n+1} := S^1 \wedge S^n$, and

$$\Omega^n_H(M) := \{\alpha \in \Omega^n(M) : d\alpha \in S^{n+1}\}.$$

Theorem
There exists a canonical extension of the graded Poisson bracket defined for $0 \leq \operatorname{order} \alpha \leq n-1$ *to*

$$\Omega^{n-1}_H(M) \otimes \Omega^n_H(M) \to \Omega^n_H(M).$$

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Remark
The second term is always semi-basic.

Example I: Classical Mechanics

Let

$$Y := Q \times \mathbb{R} \to \mathbb{R} \implies \bigwedge_2^n Y / \bigwedge_1^n Y \cong T^*Q \times \mathbb{R},$$

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so that the equations read

$$\psi^*(df) = df + \{f, \Theta\} = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q^i} \right) dt.$$

Example II: Electromagnetism

The Lagrangian $\mathcal{L} = (-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + j^{\mu}A_{\mu})d^n x$ defines the following constraints:

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Example II: Electromagnetism continued

Then, the Poisson brackets are

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Remark
The same would work for any almost-regular Lagrangian, giving a theory of Poisson brackets, and dynamics in terms of them.

Final remarks and future research

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For future research, we are interested (ongoing work) in extending the brackets presented to allow a description of the evolution of arbitrary Hamiltonian forms, thus provinding a way of looking for more general conserved quantities.

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Thank you for your attention!

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- The correspondences are not inverse of the other!

Are there non-trivial examples? Yes!