Poisson and Dirac structures in Classical Field Theories

New Advances in Contact and Symplectic Hamiltonian Dynamics

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Structure of the talk

Summary of multisymplectic field theory

Currents and conserved quantities

Graded Poisson and Dirac structuress

Summary of multisymplectic field theory

The geometric setting

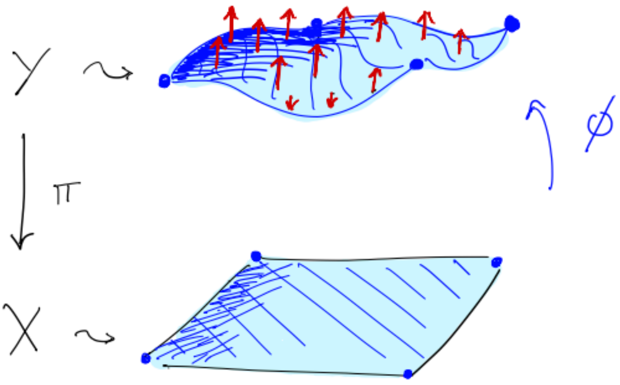
- Fields, denoted by ϕ, are sections of a fibered manifold $Y \xrightarrow{\pi} X$.
- A first order variational problem is defined through a Lagrangian density $\mathcal{L}$ on $J^1\pi_{YX}$ (which defines an n-form on X at each point), with local expression

$$\mathcal{L} = L(x^\mu, y^i, z^i_\mu)d^n x,$$

and the action can be expressed as

$$\mathcal{J}[\phi] = \int_X (j^1\phi)^*\mathcal{L}.$$

$$\xi := \frac{d}{dt}\Big|_{t=0} \phi_t$$



Stationary sections

- We define the multisymplectic form as

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}},$$

  where

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x_\mu - \left(z^i_\mu \frac{\partial L}{\partial z^i_\mu} - L \right) d^n x$$

  is the Poincaré-Cartan form.

- Stationary fields (solutions to the field equations) are characterized by

$$(j^1\phi)^* \iota_\eta \Omega_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi).$$

In coordinates,

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu} \left(\frac{\partial L}{\partial z^i_\mu} \right) = \frac{\partial L}{\partial y^i}.$$

Hamiltonian formalism I

- The extended Hamiltonian formalism takes place in

$$\bigwedge_2^n Y = \{\alpha \in \bigwedge^n T^*Y : \iota_{e_1 \wedge e_2}\alpha = 0, \text{ where } e_i \in \ker d\pi\},$$

  $\pi : Y \to X$. Locally, these forms can be expressed as

$$\alpha = p d^n x + p_i^\mu dy^i \wedge d^{n-1}x_\mu.$$

- Therefore, we have coordinates (x^μ, y^i, p_i^μ, p) representing the previous form. We have a canonical multisymplectic structure,

$$\Omega = -dp \wedge d^n x - dp_i^\mu \wedge dy^i \wedge d^{n-1}x_\mu.$$

- We can obtain the Poincaré cartan form on $J^1\pi$ as

$$\Omega_{\mathcal{L}} = \mathrm{Leg}_{\mathcal{L}}^* \Omega,$$

  where $\mathrm{Leg}_{\mathcal{L}}(x^\mu, y^i, z_\mu^i) = (x^\mu, y^i, \frac{\partial L}{\partial z_\mu^i}, L - z_\mu^i \frac{\partial L}{\partial z_\mu^i})$ is the Legendre transformation.

Hamiltonian formalism II

- We also have a reduced Hamiltonian formalism, which takes place in

$$Z^* = \bigwedge_2^n Y / \bigwedge_1^n Y,$$

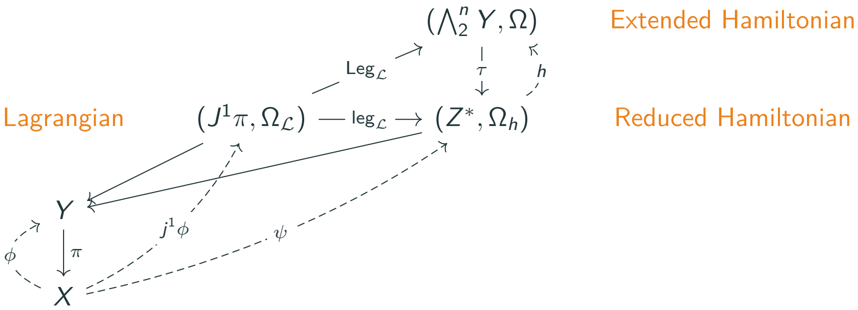
where $\{\alpha \in \bigwedge^n T^*M : \iota_e \alpha = 0, e \in \ker d\pi\}$, locally, $\alpha = p d^n x$. So we have natural coordinates (x, y^i, p_i^μ).
- Then, we obtain the reduced Legendre transformation $\mathrm{leg}_{\mathcal{L}} := \tau \circ \mathrm{Leg}_{\mathcal{L}}$, where

$$\tau : \bigwedge_2^n Y \to Z^*$$

denotes the projection.
- In order to obtain the field theory on Z^* we need a Hamiltonian section

$$h : Z^* \to \bigwedge_2^n Y.$$

Equivalence (with regular Lagrangians)



$$\phi \text{ is stationary} \Leftrightarrow (j^1\phi)^* \iota_\xi \Omega_{\mathcal{L}} = 0 \Leftrightarrow \psi^* \iota_\xi \Omega_h = 0.$$

In coordinates,

$$\frac{\partial \psi_i^\mu}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}, \; \frac{\partial \psi^i}{\partial x^\mu} = \frac{\partial H}{\partial p_\mu^i}.$$

Currents and conserved quantities

Multisymplectic manifolds

In order to work with any of the formalisms (Lagrangian, extended Hamiltonian, reduced Hamiltonian), we will work abstractly with a multisymplectic manifold, (M, ω), a manifold M together with a closed $(n+1)-$form.

We have already a lot of examples:

- $(J^1\pi, \Omega_{\mathcal{L}})$, where $\mathcal{L}$ is a Lagrangian;
- $(\bigwedge_2^n Y, \Omega)$;
- (Z^*, Ω_h), where $h : Z^* \to \bigwedge_2^n Y$ is a Hamiltonian section;
- (M, ω), where M is an orientable manifold and ω is a volume form.

Hamiltonian multivector fields and Hamiltonian forms

On every multisymplectic manifold, we have the corresponding generalization of Hamiltonian vector fields:

Definition
A multivector field $U \in \mathfrak{X}^q(M)$ is called Hamiltonian if

$$\iota_U \Omega = d\alpha,$$

where $\alpha \in \Omega^{n-q}(M)$ is called the corresponding Hamiltonian form.

We have the following:

Proposition
If U, V are Hamiltonian multivector fields of degree p, q. Then $[U, V]$ is a Hamiltonian multivector field of order $p + q - 1$. The corresponding Hamiltonian form is

$$(-1)^q \iota_{U \wedge V} \Omega.$$

Graded Poisson brackets

The previous proposition induces the following:

Definition
Let α, β be Hamiltonian forms of order $k-p, k-q$, respectively. Define their Poisson bracket

$$\{\alpha, \beta\} := (-1)^q \iota_{U \wedge V} \Omega,$$

where U, V are their respective Hamiltonian multivector fields.

Then, we have

Theorem
Modulo exact forms, the previous brackets defines a graded Lie algebra on the space of Hamiltonian forms

Question: Does it have more structure?

Properties of graded Poisson brackets

If we set $\deg\beta := k -$ order of β, then th Poisson bracket satisfies:

- *It is graded:*

$$\deg\{\alpha,\beta\} = \deg\alpha + \deg\beta;$$

- *It is graded-skew-symmetric:*

$$\{\alpha,\beta\} = -(-1)^{\deg\alpha\deg\beta}\{\beta,\alpha\};$$

- *It is local:* If $d\alpha|_x = 0$, $\{\alpha,\beta\}|_x = 0$
- *It satisfies Leibniz identity:* For $a = k$, if $\beta\wedge d\gamma \in \Omega_H^{b+c-1}(M)$, then

$$\{\beta\wedge d\gamma,\alpha\} = \{\beta,\alpha\}\wedge d\gamma + (-1)^{k-\deg\beta} d\beta\wedge\{\gamma,\alpha\};$$

- *It is invariant by symmetries:* If $X \in \mathfrak{X}(M)$ and $\pounds_X\alpha = 0$, then $\iota_X\alpha \in \Omega_H^{a-2}(M)$ and

$$\{\iota_X\alpha,\beta\} = (-1)^{\deg\beta}\iota_X\{\alpha,\beta\};$$

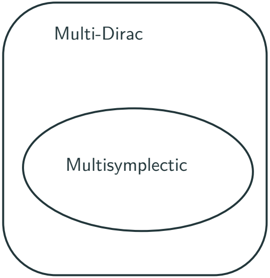
- *It satisfies graded Jacobi identity (up to an exact term):*

$$(-1)^{\deg\alpha\deg\gamma}\{\{\alpha,\beta\},\gamma\} + \text{cyclic terms} = \text{exact form}.$$

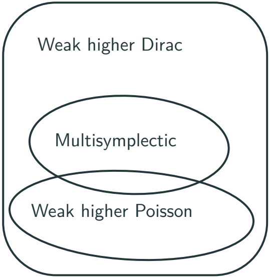
Graded Poisson and Dirac structuress

Concepts so far

Does not include Poisson!

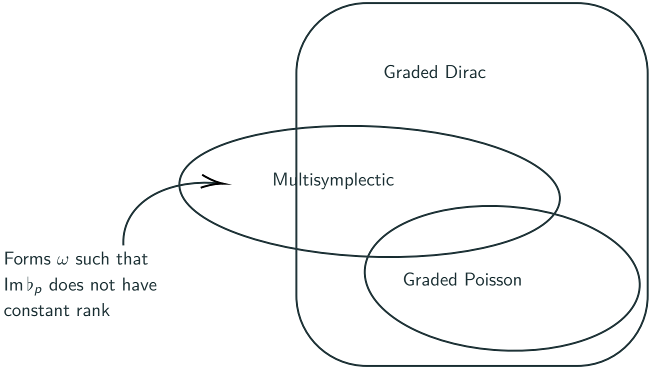


It is not graded!



Our definitions:

Includes Poisson and is graded!



Onto the definitions...

First, we look at the linearized version:

Definition
Let M be a manifold. A graded Poisson structure of order n is a tuple $(S^a, K_p, \sharp_a)$, where $S^a \subseteq \bigwedge^a M$ is a vector subbundle of forms, $K_p \subseteq \bigvee_p M (= \bigwedge^p TM)$ is a subbundle of multivectors, and

$$\sharp_a : S^a \to \bigvee_{n+1-a} M / K_{n+1-a}$$

are linear bundle maps sastifying:

- $K_p = (S^a)^{\circ,p}$, for $p \leq a$ and $K_1 = 0$.
- The maps $\sharp_a$ are *skew-symmetric*, that is,

  $$\iota_{\sharp_a(\alpha)}\beta = (-1)^{(n+1-a)(n+1-b)}\iota_{\sharp_b(\beta)}\alpha,$$

  for all $\alpha \in S^a$, $\beta \in S^b$.

And it is integrable:

- It is *integrable:* For $\alpha : M \to S^a$, $\beta : M \to S^b$ sections such that $a + b \leq 2n + 1$, and U, V multivectors of order $p = n + 1 - a$, $q = n + 1 - b$, respectively such that

$$\sharp_a(\alpha) = U + K_p,\ \sharp_b(\beta) = V + K_q,$$

we have that the $(a + b - k)$-form

$$\theta := (-1)^{(p-1)q} \pounds_U \beta + (-1)^q \pounds_V \alpha - \frac{(-1)^q}{2} d\left(\iota_V \alpha + (-1)^{pq} \iota_U \beta\right)$$

takes values in S_{a+b-k}, and

$$\sharp_{a+b-k}(\theta) = [U, V] + K_{p+q-1}.$$

Field theories as an example

Let (M, ω) be a non-degenerate multisymplectic manifold of order n, that is, the form ω must satisfy $\iota_v \omega = 0$ if and only if $v = 0$. Then, the (linear) correspondence between Hamiltonian multivector fields and forms defines a graded Poisson structure of order n:

- $$S^a = \{\iota_U \omega : U \in \bigvee_{n+1-a} M\};$$
- $$K_p = \ker_p \omega;$$
- $\sharp_a : S^a \to \bigvee_{n+1-a} / K_{n+1-a}$ is given by
$$\sharp_a(\alpha) = U + K_{n+1-a} \text{ if and only if } \iota_U \omega = \alpha.$$

In this case, $\sharp_a$ are the inverse of the $\flat_p$ (contraction) maps induced by ω.

Relationship with graded Poisson brackets

Given a graded Poisson structure on M, $(S^a, \sharp_a, K_p)$, we can define a Hamiltonian form as an $(a-1)-$form, α such that $d\alpha \in S^a$.

Definition
The Poisson bracket of Hamiltonian forms is given by

$$\{\alpha, \beta\} := (-1)^{\deg \beta} \iota_{\sharp_b(d\beta)} d\alpha$$

It satisfies all previous properties and, furthermore,

Theorem
Under some integrability conditions on the sequence of subspaces S^a, any graded Poisson structure on this family is completely characterized by the graded Poisson bracket it induces.

What about Dirac strcutures?

- We should think about graded Dirac structures as a generalization of both graded Poisson and multisymplectic;
- Non-degeneracy in graded Poisson structures is given by $K_1 = 0$, or rather,

$$\sharp_k : S^k \to TM.$$

- In order to obtain graded Dirac, we just allow for non-trivial K_1, and hence we have the "same" definition

$$\sharp_a : S^a \to \bigvee_{n+1-a} M/K_{n+1-a};$$

- Turns our that the non-degeneracy is non-essential.

Theorem
Defining

$$\{\alpha, \beta\} := (-1)^{\deg \beta} \iota_{\sharp_a d\alpha} d\beta,$$

we get a graded Poisson bracket on the space of Hamiltonian forms (defined in the same way).

Furthermore:

Theorem
Under the same integrability conditions on S^a, graded Dirac structures are also characterized by their induced graded Poisson bracket.

Remark
In this sense, these definitions respect the flavour of Poisson and Dirac geometry, giving a linearized version of brackets. In the first case, a bracket defined in a non-degenrate space of forms; and allowing for degeneracy in the second.

Final remarks and future research

- We developed the theory of Poisson bracket and tensors in classical field theories;
- What about the geometry (foliations?);
- Analogue to Lie-Poisson?;
- Reduction?

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