Coisotropic reduction in Classical Field Theories

Workshop de Jóvenes Investigadores

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Structure of the talk

The geometry of calculus of variations

- Symplectic Geometry
- Multisymplectic Manifolds
- Hamiltonian multivector fields and forms

Coisotropic submanifolds

Final remarks and future research

The geometry of calculus of variations

The geometric setting I

What to minimize/maximize? Sections!

Fixed some fibered manifold

$$Y \xrightarrow{\pi_{YX}} X \text{ with coordinates } (x^\mu, y^i) \mapsto x^\mu,$$

we want to find a section

$$\phi : X \to Y,\ (x^\mu) \mapsto (x^\mu, y^i = \phi^i(x^\mu))$$

minimizing/maximizing the functional

$$\mathcal{J}[\phi] = \int_X L\left(x^\mu, \phi^i(x^\mu), \frac{\partial \phi^i}{\partial x^\mu}, \frac{\partial^2 \phi^i}{\partial x^\mu \partial x^\nu}, \dots\right) d^n x.$$

We will focus on first order theories,

$$\mathcal{J}[\phi] = \int_X L\left(x^\mu, \phi^i(x^\mu), \frac{\partial \phi^i}{\partial x^\mu}\right) d^n x.$$

The geometric setting II

We can interpret

$$L\left(x^{\mu}, \phi^{i}(x^{\mu}), \frac{\partial \phi^{i}}{\partial x^{\mu}}\right) d^{n}x$$

as an *n*-form on the first jet bundle

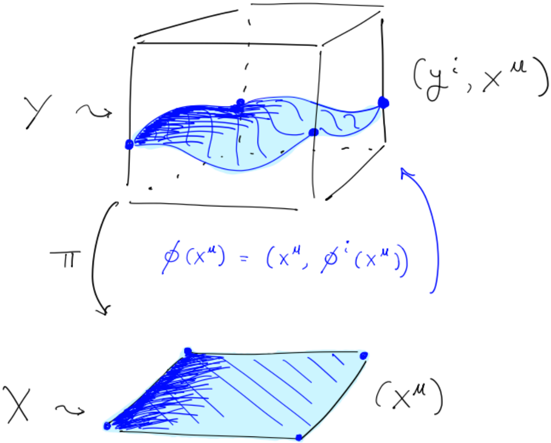
$$J^{1}\pi_{YX} \text{ with coordinates } (x^{\mu}, y^{i}, z^{i}_{\mu}).$$

We call it the Lagrangian density

$$\mathcal{L} = L(z^{\mu}, y^{i}, z^{i}_{\mu})d^{n}x.$$

We can rewrite the action as

$$\mathcal{J}[\phi] = \int_{X} (j^{1}\phi)^{*}\mathcal{L}.$$



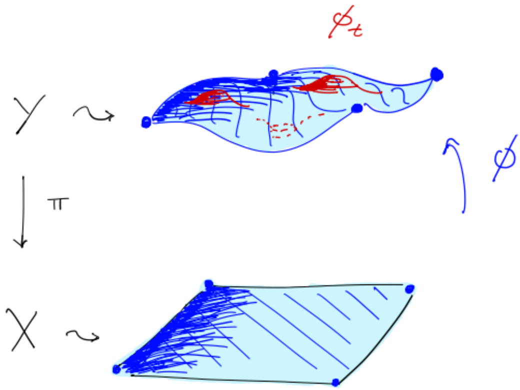
$$\mathcal{J}[\phi] = \int_X (j^1\phi)^*\mathcal{L},$$

where $\mathcal{L} \in \Omega^n(J^1\pi_{YX})$ is the Lagrangian dentisy.

The Euler-Lagrange equations I

If ϕ is a minimizer/maximizer (more generally, stationary section),

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{J}[\phi_t] = 0, \forall \text{ variation } \phi_t.$$



The Euler-Lagrange equations II

Locally, we get

$$\frac{\partial L}{\partial y^i} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial L}{\partial z^i_\mu}\right).$$

What about intrinsic Euler-Lagrange equations?

Theorem
There exists an n-form $\Theta_{\mathcal{L}}$, that can be intrinsically defined (using the geometry of $J^1\pi_{YX}$) such that a field $\phi : X \to Y$ is stationary if and only if it satifies

$$(j^1\phi)^*\iota_\eta d\Theta_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi_{YX}).$$

Locally,

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x_\mu - \left(\frac{\partial L}{\partial z^i_\mu} z^i_\mu - L\right) d^n x$$

and it is called the Poincaré-Cartan form.

Symplectic Geometry

Definition (Symplectic manifold)
A symplectic manifold is a pair (M, ω), where M is an manifold, and $\omega \in \Omega^2(M)$ is a closed, non-degenerate, 2-form.

Definition
For a subspace $i : W \hookrightarrow T_x M$, define the symplectic orthogonal as

$$W^\perp := \{v \in T_q M,\ \omega(v, w) = 0, \forall w \in W\} = \ker i^* \circ \flat.$$

$$\text{Important submanifolds} \begin{cases} \text{Lagrangian, } \ T_x L = (T_x L)^\perp \\ \\ \text{Coisotropic, } (T_x N)^\perp \subseteq T_x N \end{cases}$$

Dynamics = Lagrangian submanifolds (Weinstein's creed)

$$(M, \omega) \text{ symplectic} \to (TM, \widetilde{\omega}) \text{ symplectic},$$

$$\widetilde{\omega} = \flat_\omega^* \omega_M;\ \flat_\omega : TM \to T^*M \text{ (contraction)}$$

Definition

- Hamiltonian vector field: $X_H \in \mathfrak{X}(M)$, $(H \in C^\infty(M))$ such that

$$\iota_{X_H}\omega = dH.$$

- Locally Hamiltonian vector field: $X \in \mathfrak{X}(M)$ such that

$$d\iota_X\omega = 0.$$

Theorem
A vector field $X : M \to TM$ is locally Hamiltonian if and only if it defines a Lagrangian submanifold of $(TM, \widetilde{\omega})$.

Coisotropic reduction

Given a coisotropic submanifold $i : N \hookrightarrow M$, the distribution

$$x \mapsto (T_x N)^\perp$$

is regular and involutive. Therefore, it arises from a maximal foliation $\mathcal{F}$. Then,

Theorem
If $N/\mathcal{F}$ admits a smooth manifold structure such that $\pi : N \to N/\mathcal{F}$ defines a submersion ($N/\mathcal{F}$ is a quotient manifold), then there is an unique symplectic form ω_N on $N/\mathcal{F}$ such that

$$\pi^* \omega_N = i^* \omega.$$

Furthermore, if L is a Lagrangian submanifold in M that has clean intersection with N, $\pi(L \cap N)$ is a Lagrangian submanifold in $(N/\mathcal{F}, \omega_N)$

Poisson brackets

Definition
(M, ω) symplectic manifold, $f, g \in C^\infty(M)$.

$$\text{Poisson bracket: } \{f, g\} = \omega(X_f, X_g).$$

- Jacobi indentity

$$\{f, \{g, h\}\} + \text{cycl.} = 0,$$

- Leibniz indentity

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

Theorem
A submanifold $N \hookrightarrow M$ is coisotropic if and only if

$$I_N = \{f \in C^\infty(M) : df = 0 \text{ on } N\}$$

defines a Poisson subalgebra of $(C^\infty, \{\cdot, \cdot\})$.

Multisymplectic Manifolds

Multisymplectic Manifolds

Definition
A multisymplectic manifold of order k is a pair (M, ω), where M is a smooth manifold, and ω is a closed $(k+1)-$form.

No non-degeneracy required

Definition
For $W \subseteq T_x M$, and $1 \leq j \leq k$ define the multisymplectic orthogonal as

$$W^{\perp,j} := \{v \in T_x M : \ \iota_{v \wedge w_1 \wedge \cdots w_j}\omega = 0, \ \forall w_1, \ldots, w_j \in W\}.$$

$$\text{Important submanifolds} \begin{cases} j - \text{Lagrangian}, \ T_x L + \ker \flat_1 = (T_x L)^{\perp,j} \\ \\ j - \text{Coisotropic}, \ (T_x N)^{\perp} \subseteq T_x N + \ker \flat_1 \end{cases}$$

Hamiltonian multivector fields and forms

Hamiltonian multivector fields and forms

The generalization of the concept of Hamiltonian vector field is the following:

Definition (Hamiltonian multivector field)
Given a multisymplectic manifold of order k, (M, ω), we say that a multivector field $U \in \mathfrak{X}^q(M)$ is Hamiltonian if

$$\iota_U \omega = d\alpha,$$

where $\alpha \Omega^{k-q}(M)$. α is called Hamiltonian form.

We have an equivalence:

$$\begin{array}{ccccc}
\widehat{\mathfrak{X}}^1_H(M) & \cdots & \widehat{\mathfrak{X}}^q_H(M) & \cdots & \widehat{\mathfrak{X}}^k_H(M) \\
\Big\updownarrow \flat_1 & & \Big\updownarrow \flat_q & & \Big\updownarrow \flat_k \\
\widehat{\Omega}^{k-1}_H(M) & \cdots & \widehat{\Omega}^{k-q}_H(M) & \cdots & \widehat{\Omega}^0_H(M)
\end{array} .$$

Dynamics = Lagrangian submanifolds

$$(M,\omega) \text{ multisymplectic} \to \left(\bigvee_q M, \widetilde{\Omega}^q\right) \text{ multisymplectic}$$

$$\widetilde{\Omega}_q = \flat_q^* \Omega_M^{k+1-q},\ \flat_q : \bigvee_q M \to \bigwedge^{k+1-q} M \ \text{(contraction)}$$

Definition

- Locally Hamiltonian multivector field: $U : M \to \bigvee_q M$ such that

$$d\iota_U\omega = 0.$$

Theorem
A multivector field $U : M \to \bigvee_q M$ is locally Hamiltonian if and only if it defines a $(k+1-q)-$Lagrangian submanifold in $\left(\bigvee_q M, \widetilde{\Omega}^q\right)$

Coisotropic submanifolds

Coisotropic reduction

Given a k-coisotropic submanifold $i : N \hookrightarrow M$, we have

Proposition
The distribution $x \mapsto (T_x N)^{\perp,k} \cap T_x N \subseteq T_x N$ *is involutive.*

Thus, when it is regular, it arises from a foliation $\mathcal{F}$.

Theorem
When $N/\mathcal{F}$ *admits a smooth manifold structure such that the projection* $\pi : N \to N/\mathcal{F}$ *defines a submersion (*$N/\mathcal{F}$ *is a quotient manifold), there exists an unique multisymplectic form* ω_N *on* $N/\mathcal{F}$ *such that*

$$\pi^* \omega_N = i^* \omega.$$

What about projection of Lagrangian submanifolds?

Multisymplectic manifolds of type (k, r)

Definition
Let L be a manifold and $\mathcal{E}$ be a regular distribution on L. Define:

$$\bigwedge_r^k L = \{\alpha \in \bigwedge^k L : \ \iota_{e_1 \wedge \cdots \wedge e_r} \alpha = 0, \forall e_1, \ldots, e_r \in \mathcal{E}\}.$$

$$\left(\bigwedge_r^k L, \Omega_L \right) \text{ is a multisymplectic manifold}$$

Definition
A multisymplectic manifold of type (k, r) $(M, \omega, W, \mathcal{E})$ is a multisymplectic manifold (M, ω) that is locally multisymplectomorphic to $\bigwedge_r^k L$.

$$W \sim \text{ vertical distribution}$$

An example of coisotropic reduction

Let L be a smooth manifold, $i : Q \subseteq L$ be a submanifold, and $\mathcal{E}$ be a regular distribution. Then,

Proposition
$N := \bigwedge_r^k L\big|_Q$ defines a k-coisotropic submanifold.

Theorem
For $N = \bigwedge_r^k L\big|_Q$, where $TQ \cap \mathcal{E}$ has constant rank,

$$N/\mathcal{F} \cong \bigwedge_r^k Q.$$

Projection of Lagrangian submanifolds (example)

[fragile] An important class of Lagrangian submanifold are given by closed forms, since horizontal k-Lagrangian submanifolds are locally the image of closed forms.

$$\begin{cases} N = \bigwedge_r^k L\big|_Q, \\ \\ \alpha : L \to \bigwedge_r^k L. \end{cases} \xrightarrow{\text{Coisotropic reduction}} \begin{cases} N/\mathcal{F} = \bigwedge_r^k Q, \\ \\ i^*\alpha : Q \to \bigwedge_r^k Q. \end{cases}$$

Theorem
In our example, k-Lagrangian submanifolds transversal to the vertical distribution reduce to k-Lagrangian submanifolds.

Local characterization of vertical coisotropic submanifolds

Definition
Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type (k, r). A submanifold $i : N \hookrightarrow M$ is called vertical if $W|_N \subseteq TN$.

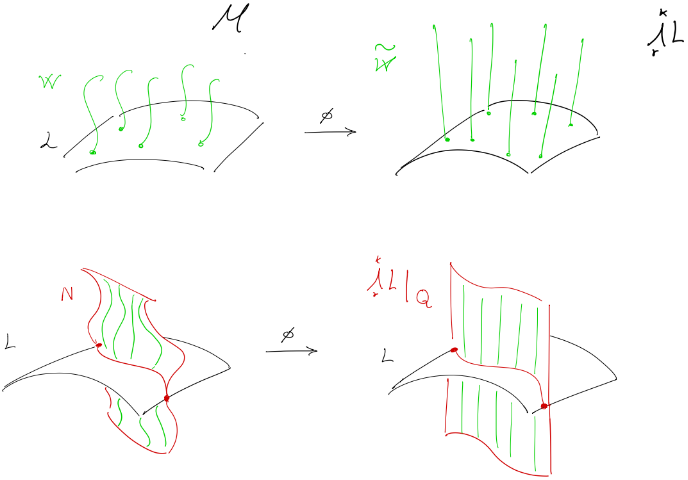
Theorem
Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type (k, r), $i : N \hookrightarrow M$ be a vertical k-coisotropic submanifold, and $j : L \hookrightarrow M$ be a k-Lagrangian submanifold complementary to W. Then there is a neighborhood U of L in M, a submanifold $Q \hookrightarrow L$, a neighborhood V of L in $\bigwedge_r^k L$, and a multisymplectomorphism

$$\phi : U \to V$$

satisfying

a) ϕ *is the identity on* L;
b) $\phi(N \cap U) = \bigwedge_r^k L\big|_Q \cap V$.

Idea of the proof



Lagrangian submanifold projection

This local characterization allows us to prove:

Theorem
Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type (k, r), $i : N \hookrightarrow M$ be a vertical k-coisotropic submanifold, and $j : L \hookrightarrow M$ be k-Lagrangian submanifold complementary to W. If $TN/W \cap \mathcal{E}$ has constant rank, so does $(TN)^{\perp,k}$ and we have that, denoting by $\pi : N \to N/\mathcal{F}$ the canonical projection, $\pi(L \cap N)$ is k-Lagrangian in (N, ω_N).

A general result is not possible, since we can easily find counterexamples.

Poisson bracket

Definition
Given two Hamiltonian forms $\alpha \in \Omega^{l_1}(M), \beta \in \Omega^{l_2}(M)$ on (M, ω),

$$\text{Poisson bracket: } \{\alpha, \beta\} := (-1)^{k-1-l_2} \iota_{X_\alpha \wedge X_\beta} \omega,$$

$$\iota_{X_\alpha} \omega = d\alpha, \ \iota_{X_\beta} \omega = d\beta.$$

- Well-defined (independent of the choice of X_α, X_β),
- Modulo closed-forms, it defines a graded Lie algebra on Hamiltonian forms
$$(-1)^{\deg \widehat{\alpha} \deg \widehat{\gamma}} \{\widehat{\alpha}, \{\widehat{\beta}, \widehat{\gamma}\}\} + \text{cycl.} = 0,$$
for
$$\widehat{\alpha} := \alpha + (\text{closed forms}), \ \deg \widehat{\alpha} := k - 1 - \text{order}(\alpha).$$

Poisson bracket

- Restricts to a Lie bracket on

$$\widehat{\Omega}_H^{k-1}(M) := (\text{Hamiltonian } (k-1)-\text{forms})\big/(\text{closed } (k-1)-\text{forms})$$

Proposition
A k-coisotropic submanifold $i : N \hookrightarrow M$ defines a Lie subalgebra

$$I_N = \{\widehat{\alpha} \in \widehat{\Omega}_H^{k-1}(M),\ i^* d\alpha = 0\}$$

of the Lie algebra $\widehat{\Omega}_H^{k-1}(M)$.

Final remarks and future research

Final remarks and future research

- We gave an interpretation of dynamics as Lagrangian submanifolds.
- We proved a coisotropic reduction theorem in a particular class of multisymplectic manifolds.
- For future research we have proposed the following:
  - Apply the results obtained to Field Theories (regularization, constraint analysis, etc)
  - Extend these results to multicontact geometry for the study of dissipative fields.
  - Connect these ideas to higher analogues of Dirac structures (giving a unified framework for both the Lagrangian and Hamiltonian formulation of Field Theory).
  - Explore the induced geometry by the Poisson brackets (generalization of Poisson tensor?)

References

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