Graded Brackets in Classical Field Theory: Conservative and Dissipative

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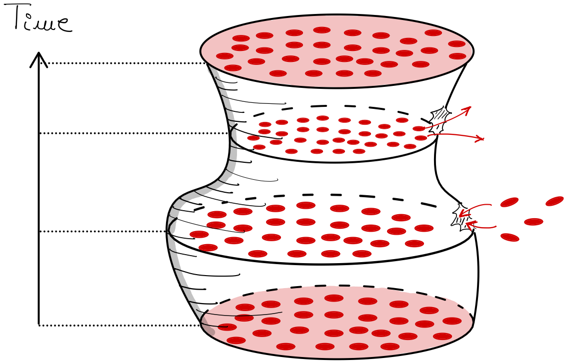
Introduction to the problem

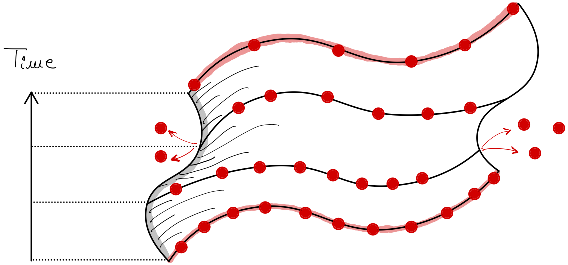
Conserved quantities in field theories I



For $\alpha \in \Omega^{n-1}(M)$:

$$\int_{X_{t_1}} \alpha = \int_{X_{t_2}} \alpha - \int_{\partial X \times [t_1, t_2]} \alpha \iff \mathrm{d}\alpha = 0.$$

Conserved quantities in field theories II



For $\alpha \in \Omega^a(M)$:

$$\int_{X_{t_1}} \alpha = \int_{X_{t_2}} \alpha - \int_{\partial X \times [t_1, t_2]} \alpha \iff \mathrm{d}\alpha = 0.$$

Proposed problem

Problem (that we would like to solve): Find all conserved quantities $\sim$ Find forms that are closed on solutions.

Problem (that we solve): Determine evolution of forms via some bracket:

$$\psi^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}\,,$$

where ψ is a solution of the corresponding PDE and α is **some** *a*-form.

Previous work:

1. Igor V. Kanatchikov. **"Canonical Structure of Classical Field Theory in the Polymomentum Phase Space".** In: *Rep. Math. Phys.* **41**.1 (1998), pp. 49–90
2. Miguel Á. Berbel and Marco Castrillón-López. **"Poisson–Poincaré Reduction for Field Theories".** In: *J. Geom. Phys.* **191** (2023), p. 104879
3. François Gay-Balmaz, Juan C. Marrero, and Nicolás Martínez-Alba. **"A New Canonical Affine Bracket Formulation of Hamiltonian Classical Field Theories of First Order".** In: *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **118**.3 (2024), p. 103

Poisson bracket of functions

The Poisson bracket on the **phase space** T^*Q is defined as

$$\dot{f} = \frac{\partial f}{\partial t} + \{f, H\} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i}\,,$$

where H is the Hamiltonian determining the dynamics.

Then, a time independent observable $f \in C^\infty(M)$ is a conserved quantity if it is in **involution** $\{f, H\} = 0$.

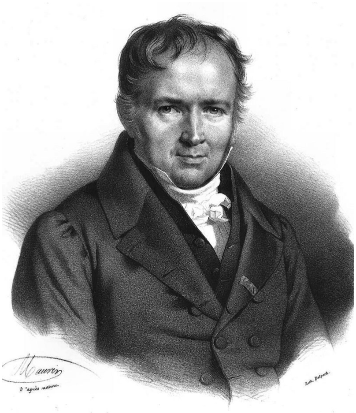
On an arbitrary **symplectic manifold** (M, ω), where $\omega \in \Omega^2(M)$, $\mathrm{d}\omega = 0$, non-degenerate, is:

$$\forall f \in C^\infty(M), \exists! X_f \in \mathfrak{X}(M), \iota_{X_f}\omega = \mathrm{d}f\,,$$

$$\{f, g\} := \omega\left(X_f, X_g\right) = X_g(f) = -X_f(g)\,.$$

Poisson Theorem

Poisson Theorem: If $0 = \{f, H\} = \{g, H\}$, we have $\{\{f, g\}, H\} = 0$.



Siméon Denis Poisson 1781-1840

Jacobi theorem

Jacobi Theorem:

$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0.$

Carl Gustav Jacob Jacobi 1804-1851

Geometric stage and algebraic structure of observables

General setup for field equations

Let $\tau : M \longrightarrow X$ denote a fibered manifold, $n = \dim X$. Let:

(i) $\alpha_1, \ldots, \alpha_k \in \Omega^{n-1}(M)$ be **semi-basic forms** (representing observables).

(ii) $\beta_1, \ldots, \beta_k \in \Omega^n(M)$ be **semi basic forms** (representing evolution of observables).

Semi-basic: Locally, a semi-basic form is written as $f \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_a}$, $f \in C^\infty(M)$.

We deal with partial differential equations with the following structure*:

$$\psi^*(\mathrm{d}\alpha_i) = \beta_i \circ \psi \,, \text{where } \psi : X \to M \text{ is a subsection.}$$

Examples (equations) I

Hamilton's equations:

(a) Fiber bundle: $T^*Q \times \mathbb{R} \to \mathbb{R}$.

(b) Semi-basic forms: q^i, p_i, t.

(c) Basic forms: $\frac{\partial H}{\partial p_i}\mathrm{d}t\,, -\frac{\partial H}{\partial q^i}\mathrm{d}t$, $\mathrm{d}t$.

And we obtain the equations:

$$\dot{q}^i\mathrm{d}t = \frac{\partial H}{\partial p_i}\mathrm{d}t\,, \quad \dot{p}_i\mathrm{d}t = -\frac{\partial H}{\partial q^i}\mathrm{d}t\,.$$

Examples (equations) II

Notation: $\mathrm{d}^n x = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$,
$\mathrm{d}^{n-1}x_\mu = (-1)^\mu \mathrm{d}x^0 \wedge \cdots \wedge \widehat{\mathrm{d}x^\mu} \wedge \cdots \wedge \mathrm{d}x^{n-1}$.

Hamilton–De Donder–Weyl equations:

(a) Fiber bundle: (covariant phase space) $\bigwedge_2^n Y / \bigwedge_1^n Y \to X$.

(b) Semi-basic forms: $y^i \mathrm{d}^{n-1}x_\mu$, $p_i^\mu \mathrm{d}^{n-1}x_\mu$, $\frac{1}{n} x^\mu \mathrm{d}^{n-1}x_\mu$.

(c) Basic forms: $\frac{\partial H}{\partial p_i^\mu}\mathrm{d}^n x$, $-\frac{\partial H}{\partial y^i}\mathrm{d}^n x$, $\mathrm{d}^n x$.

And we obtain the equations:

$$\frac{\partial y^i}{\partial x^\mu}\mathrm{d}^n x = \frac{\partial H}{\partial p_i^\mu}\mathrm{d}^n x\,, \quad \frac{\partial p_i^\mu}{\partial x^\mu}\mathrm{d}^n x = -\frac{\partial H}{\partial y^i}\mathrm{d}^n x\,.$$

Examples (equations) III

Yang–Mills equations:

(a) Fiber bundle: $\operatorname{Im}\operatorname{leg}_{\mathcal{L}} \to X$.

(b) Semi-basic forms:
$A^i_\mu \mathrm{d}^{n-1}x_\nu - A^i_\nu \mathrm{d}^{n-1}x_\mu,\ F^{\mu\nu}_i \mathrm{d}^{n-1}x_\nu,\ \frac{1}{n}x^\mu \mathrm{d}^{n-1}x_\mu.$

(c) Basic forms:
$\left(-F^i_{\mu\nu} + f^i_{jk}A^j_\nu A^k_\mu\right)\mathrm{d}^n x,\ \left(-f^i_{jk}F^{\mu\nu}_i A^k_\mu\right)\mathrm{d}^n x,\ \mathrm{d}^n x.$

And we obtain the equations:

$$\left(\frac{\partial A^i_\mu}{\partial x^\nu} - \frac{\partial A^i_\nu}{\partial x^\mu}\right)\mathrm{d}^n x = \left(-F^i_{\mu\nu} + f^i_{jk}A^j_\nu A^k_\mu\right)\mathrm{d}^n x\,,$$

$$\frac{\partial F^{\mu\nu}_i}{\partial x^\mu}\mathrm{d}^n x = -f^i_{jk}F^{\mu\nu}A^k_\mu \mathrm{d}^n x\,.$$

Some comments

(i) **Mostly any** PDE fits into the description described earlier. However, we will require further structure in the sequel.

(ii) So far we have given a local description, and what we will describe will work globally by requiring the previous picture to hold patch-wise on any fibered manifold $\tau\colon M \to X$. More particularly, we work with the subbundle

$$S^n := \langle \mathrm{d}\alpha_1, \ldots, \mathrm{d}\alpha_k \rangle \subseteq \bigwedge^n T^*M \,,$$

and a map $\beta\colon S^n \longrightarrow \bigwedge T^*X$.

Algebraic structure of observables

The observables $\alpha_1 \ldots, \alpha_k \in \Omega^{n-1}(M)$ allow us to define the space of **Hamiltonian forms**:

$$\Omega^{n-1}_H(M) := \{\alpha \in \Omega^{n-1}(M) \colon \mathrm{d}\alpha \in \langle \mathrm{d}\alpha_1, \ldots \mathrm{d}\alpha_k \rangle\}\,.$$

Assumption 1: There is a bracket $\{\cdot,\cdot\}$ on the space of Hamiltonian forms satisfying the following properties:

(i) It is *skew-symmetric*: $\{\alpha,\beta\} = -\{\beta,\alpha\}$.

(ii) It satisfies the Jacobi identity up to an exact term:

$$\{\alpha,\{\beta,\gamma\}\} + \{\beta,\{\gamma,\alpha\}\} + \{\gamma,\{\alpha,\beta\}\} = \text{exact form}\,.$$

(iii) It vanishes on closed forms: $\mathrm{d}\alpha = 0 \implies \{\alpha,\beta\} = 0$.

(iv) There is a correspondence $\alpha \mapsto X_\alpha$ such that $\{\alpha,\beta\} = \iota_{X_\beta}\mathrm{d}\alpha$.

Parenthesis: How to obtain such a bracket?

Suppose we work with a **first order** variational problem on the sections of a configuration fiber bundle $\pi\colon Y \to X$ given by a Lagrangian density

$$\mathcal{L}\colon J^1\pi \to \bigwedge^n T^*X\,, \quad \mathcal{L}(\phi) = L(x^\mu, \phi^i, \frac{\partial \phi^i}{\partial x^\mu})\mathrm{d}^n x\,.$$

The action is

$$\mathcal{J}[\phi] = \int_X \mathcal{L}(\phi)\,.$$

We then find the **multisymplectic form** $\Omega_{\mathcal{L}} \in \Omega^{n+1}(J^1\pi)$, which is closed and allows to define the bracket:

$$\forall \alpha \in \Omega^{n-1}_H(J^1\pi), \exists X_\alpha \in \mathfrak{X}(M), \iota_{X_\alpha}\Omega_{\mathcal{L}} = \mathrm{d}\alpha\,,$$

$$\{\alpha, \beta\} := \iota_{X_\alpha \wedge X_\beta}\Omega_{\mathcal{L}}\,.$$

Examples (algebraic structure)

(i) **Hamilton equations**: Non vanishing brackets:

$$\{q^i, p_j\} = \delta^i_j \,.$$

(ii) **Hamilton–De Donder–Weyl equations**: Non vanishing brackets:

$$\{y^i \mathrm{d}^{n-1}x_\mu, p^\nu_j \mathrm{d}^{n-1}x_\nu\} = \delta^i_j \mathrm{d}^{n-1}x_\nu \,.$$

(iii) **Yang–Mills equations**: Non vanishing brackets:

$$\{A^i_\mu \mathrm{d}^{n-1}x_\nu - A^i_\nu \mathrm{d}^{n-1}x_\mu, F^{\alpha\beta}_j \mathrm{d}^{n-1}x_\beta\} = \delta^i_j \delta^{\alpha\beta}_{\mu\nu} \mathrm{d}^{n-1}x_\beta \,.$$

Hamiltonian forms of arbitrary order I

Definition
We say that a form $\alpha \in \Omega^a(M)$ is **special Hamiltonian** if there is a semi-basic form $\beta \in \Omega^{a+1}(M)$ such that $\psi^*(\mathrm{d}\alpha) = \beta \circ \psi$, for every solution of the equations. The space of special Hamiltonian a-forms is denoted by $\widetilde{\Omega}^a_H(M)$.

Remark
If $\alpha \in \Omega^{n-1}(M)$ is special Hamiltonian, $\alpha \in \Omega^{n-1}_H(M)$.

Theorem (de León, I.L. 2025)
$\alpha \in \Omega^a(M)$ is special Hamiltonian if and only if $\alpha \wedge \varepsilon \in \Omega^{n-1}_H(M)$, for every closed and basic $\varepsilon \in \Omega^{n-1-a}(M)$.

Hamiltonian forms of arbitrary order II

Definition
A form $\alpha \in \Omega^a_H(M)$ is called **Hamiltonian** if $\mathrm{d}\alpha \in \iota_{\bigwedge^{n-(a+1)} TM}\langle \mathrm{d}\alpha_1, \dots, \mathrm{d}\alpha_k \rangle$. We denote by $\Omega^a_H(M)$ the space of Hamiltonian a-forms.

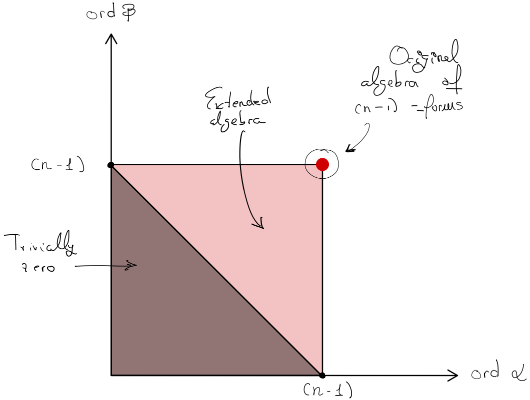
Proposition
$\widetilde{\Omega}^a_H(M) \subseteq \Omega^a_H(M)$

Theorem (de León, I.L. 2025)
There is an unique induced graded Poisson bracket

$$\Omega^a_H(M) \otimes \Omega^b_H(M) \to \Omega^{a+b-(n-1)}_H(M)$$

that maintains the properties of the original bracket of $(n-1)$-forms. Furthermore, special Hamiltonian forms define a subalgebra.

Summary

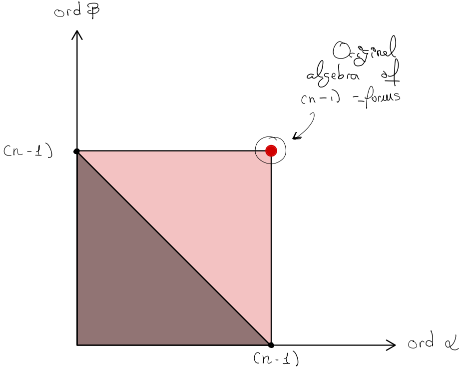

ord β
Original algebra of (n − 1) − forms
Extended algebra
(n − 1)
Trivially zero
(n − 1)
ord α

Hamiltonians and extensions of brackets

Domain of definition of current brackets

$$\begin{cases} \psi^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\} \\ \deg\{\alpha, \mathcal{H}\} = \deg\alpha + \deg\mathcal{H} - (n-1) \end{cases} \implies \deg\mathcal{H} = n.$$

But:



First extension of the brackets I

Theorem (de León, I.L. 2025)
There exists an unique extension of $\{\cdot, \cdot\}$

$$\Omega^{n-1}_H(M) \otimes \Omega^a_H(M)[1] \to \Omega^a_H(M)[1]$$

for arbitrary $a \geq 0$ *that satisfies the properties of* $\{\cdot, \cdot\}$.

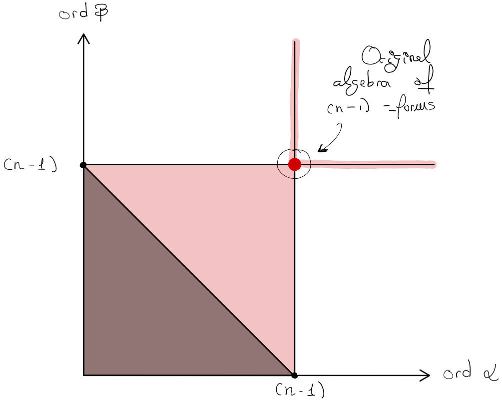
Assumption 2: There exists a form $\mathcal{H} \in \Omega^n(M)[1]$, the Hamiltonian, such that

$$\psi^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\} \,,$$

for every solution ψ and $\alpha \in \Omega^{n-1}_H(M)$.

First extension of the brackets II

Current domain of definition:



Examples (Hamiltonians)

(i) **Hamilton equations**:

$$\mathcal{H} = H\mathrm{d}t - p_i \mathrm{d}q^i \,.$$

(ii) **Hamilton–De Donder–Weyl equations**:

$$\mathcal{H} = H\mathrm{d}^n x - p_i^\mu \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu \,.$$

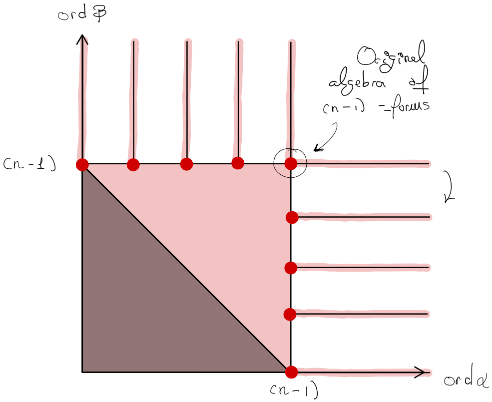
(iii) **Yang–Mills equations**

$$\mathcal{H} = \left(-\frac{1}{4}F_i^{\mu\nu}F^i_{\mu\nu} + \frac{1}{2}f^i_{jk}F_i^{\mu\nu}A^j_\mu A^k_\nu\right) d^n x - F_i^{\mu\nu}\mathrm{d}A^i_\mu \wedge \mathrm{d}^{n-1}x_\nu \,.$$

Hamiltonian = Poincaré–Cartan form

Final extension of the bracket I

Question: Can we interpret $\psi^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}$, for arbitrary $\alpha \in \Omega^a_H(M)$?



Final extensions of the brackets II

Problem: There is no unique extension of the bracket to order $a < n-1$.

Nevertheless,

Theorem (de León, I.L. 2025)
There is a bijective correspondence between the possible extensions of the bracket and affine maps

$$\gamma : \{\mathit{Hamiltonians}\} \to \{\mathit{Ehresmann\ connections\ on}\ \tau : M \to X\}$$

such that $\gamma(\mathcal{H})$ solves the Hamilton–De Donder–Weyl equations of $\mathcal{H}$, for every $\mathcal{H}$.

Final extension of the bracket III

Corollary
Let γ be such a map, $\mathcal{H}$ be a Hamiltonian, and ψ be an integral section of $\gamma(\mathcal{H})$. Then

$$\psi^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}_\gamma \,,$$

for every $\alpha \in \Omega^a_H(M)$.

Corollary
Let α be a special Hamiltonian form. Then, the bracket $\{\alpha, \mathcal{H}\}_\gamma$ is independent of extension $\{\cdot, \cdot\}_\gamma$, for every Hamiltonian $\mathcal{H}$.

Theorem (Poisson theorem)
Special Hamiltonian forms that are closed on solutions close a subalgebra under the graded Poisson bracket.

Technical remarks

The construction was based on a generalization of the $\sharp$ mapping associated to a graded Poisson bracket. In particular, we generalized the techniques employed in

1. Peter W. Michor. **"A Generalization of Hamiltonian Mechanics".** In: *J. Geom. Phys.* **2**.2 (1985), pp. 67–82
2. Janusz Grabowski. **"Z-Graded Extensions of Poisson Brackets".** In: *Rev. Math. Phys.* **09**.01 (1997), pp. 1–27

to extend the brackets.

Into a theory of brackets in dissipative field theories

The Herglotz variational principle for fields I

We have presented the previous questions in the usual framework of variational problems. However, there is a growing interest in more general variational problems, such as those involving **Lagrangians dependent on the action**.

1. Matheus J. Lazo et al. **"An Action Principle for Action-Dependent Lagrangians: Toward an Action Principle to Non-Conservative Systems".** In: *J. Math. Phys.* **59**.3 (2018). 10.1063/1.5019936, p. 032902
2. Jordi Gaset et al. **"The Herglotz Variational Principle for Dissipative Field Theories".** In: *Geom. Mech.* **01**.02 (2024). 10.1142/S2972458924500060, pp. 153–178

The Herglotz variational principle for fields II

Let $\pi\colon Y \to X$be a fiber bundle, and let $\mathcal{L}\colon J^1\pi \times_X \bigwedge^{n-1} T^*X \to \bigwedge^n T^*X$ be a **Lagrangian dependent on the action**. Locally,

$$\mathcal{L} = L(x^\mu, \phi^i, \frac{\partial \phi^i}{\partial x^\mu}, s^\mu)\mathrm{d}^n x.$$

We look for fields $\phi^i(x^\mu)$ minimizing the following problem:

$$\text{Look for } \zeta^\mu \text{ satisfying } \frac{\partial \zeta^\mu}{\partial x^\mu} = L\left(x^\mu \phi^i, \frac{\partial \phi^i}{\partial x^\mu}, \zeta^\mu\right),$$

$$\text{Minimize } \int_X L\mathrm{d}^n x.$$

The Herglotz equations of motion and multicontact geometry

The equations obtained with the previous variational principle are the **Herglotz equations of motion**:

$$\frac{\partial L}{\partial \phi^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial L}{\partial(\partial_\mu \phi^i)}\right) + \frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial(\partial_\mu \phi^i)} = 0\,,$$

$$\frac{\partial s^\mu}{\partial x^\mu} = L\,.$$

Intrinsically, these are described by the **multicontact form** $\Theta_{\mathcal{L}} \in \Omega^n(J^1\pi)$.

$$\Theta_{\mathcal{L}} = \left(L - \partial_\mu \phi^i \frac{\partial L}{\partial(\partial_\mu \phi^i)}\right)\mathrm{d}^n x + \frac{\partial L}{\partial(\partial_\mu \phi^i)}\mathrm{d}\phi^i \wedge \mathrm{d}^{n-1}x_\mu + \mathrm{d}s^\mu \wedge \mathrm{d}^{n-1}x_\mu\,.$$

Graded Jacobi brackets I

Let $M := J^1\pi \times_X \bigwedge^{n-1} T^*X$. Borrowing from contact geometry:

We say that a multivector field $X \in \mathfrak{X}^p(M)$ (sum of decomposable $X_1 \wedge \cdots \wedge X_p$), is a **infinitesimal conformal transformation** of $\Theta_{\mathcal{L}}$ if

$$\pounds_U \Theta_{\mathcal{L}} = \mathrm{d}\iota_U \Theta_{\mathcal{L}} + (-1)^{p-1}\iota_U \mathrm{d}\Theta_{\mathcal{L}} = \iota_V \Theta_{\mathcal{L}} \,,$$

for certain $V \in \mathfrak{X}^{p-1}(M)$ called the **conformal factor**.

Graded Jacobi brackets II

Definition
A form $\alpha \in \Omega^a(M)$ is called **conformal Hamiltonian** if there is a conformal multivector field U such that $\alpha = \iota_U \Theta$. The space of conformal Hamiltonian forms is denoted by $\Omega^a_H(M)$.

Theorem (de Léon, I.L., Rivas)
Let U_1 and U_2 be conformal infinitesimal transformations of $\Theta_{\mathcal{L}}$. Then, $\iota_{[U_1,U_2]}\Theta_{\mathcal{L}}$, where $[U_1, U_2]$ is the Schouten–Nijenhuis bracket, only depends on the values of $\iota_{U_1}\Theta_{\mathcal{L}}$ and $\iota_{U_2}\Theta_{\mathcal{L}}$.

Definition
Then, $\{\alpha_1, \alpha_2\} := -\iota_{[U_1,U_2]}\Theta_{\mathcal{L}}$ is called the **graded Jacobi bracket** of $\alpha_i := \iota_{U_i}\Theta_{\mathcal{L}}$.

Graded Jacobi brackets III

Defining $\deg_H \alpha := (n-1) - \deg \alpha$, the bracket has the following properties:

Theorem (de León, I.L., Rivas)
The graded Jacobi bracket defines an operation

$$\Omega_H^a(M) \otimes \Omega_H^b(M) \xrightarrow{\{\cdot,\cdot\}} \Omega_H^{a+b-(n-1)}(M)$$

satisfying the following properties:

(i) *It is* **graded-skew-symmetric**
$\{\alpha, \beta\} = -(-1)^{\deg_H \alpha \deg_H \beta}\{\beta, \alpha\}$;

(ii) *It satisfies the* **graded Jacobi identity***:*

$$(-1)^{\deg_H \alpha \deg_H \gamma}\{\alpha, \{\beta, \gamma\}\} + \mathit{cycl.} = 0\,.$$

Conclusions and references

Final remarks and remaining questions

(i) The previous theoretical results seem to indicate that the subalgbra of special Hamiltonian forms is of high relevance to a particular field theory. We would like to compute these subalgebras for several almost regular Lagrangians to further study these classical field theories.

(ii) (In progress) We would also like to investigate the relation between these extensions and the instantaneous split formalism.

(iii) It is also interesting to investigate the implications of this algebraic structure in the study of momentum maps and reduction, employing the graded brackets.

(iv) Is it possible to extend all of the results in the conservative scenario to the dissipative scenario?

Main references

1. Manuel de León and Rubén Izquierdo-López. **"Graded Poisson and Graded Dirac Structures".** In: *J. Math. Phys.* **66**.2 (2025). 10.1063/5.0243128, p. 022901
2. Manuel de León and Rubén Izquierdo-López. ***A Description of Classical Field Equations Using Extensions of Graded Poisson Brackets*.** 10.48550/arXiv.2507.04743. 2025
3. Manuel de León, Rubén Izquierdo-López, and Xavier Rivas. ***Brackets in Multicontact Geometry and Multisymplectization*.** 10.48550/arXiv.2505.13224. 2025

Thank you for your attention!

The graded Dirac structure of an almost regular Lagrangian

Graded Dirac structures I

Definition
A (fibered) **graded Dirac structure** on a fibered manifold $\tau \colon M \to X$ is a tuple $(S^a, \sharp_a)$, where

(i) S^a, $a = 1, \dots, n$ is a subbundle of $\bigwedge^a T^*M$, and is composed of $(a-1)$-horizontal forms.

(ii) $\sharp_a$ are vector bundle mappings

$$\begin{tikzcd} S^a \arrow[r, "\sharp_a"] \arrow[d] & \bigwedge^{n+1-a} TM/K_{n+1-a} \arrow[dl, bend left] \\ M & \end{tikzcd}$$

Graded Dirac structures II

These maps $\sharp_a$ are required to satisfy the following

(a) They are skew-symmetric:
$\iota_{\sharp_a(\alpha)}\beta = (-1)^{(n+1-\deg\alpha)(n+1-\deg\beta)}\iota_{\sharp_\beta}\alpha$

(b) They are **integrable** (involutive with respect to a generalization of the Courant bracket).

$$
\begin{tikzcd}
S^a \arrow[r, "\sharp_a"] \arrow[d] & \bigwedge^{n+1-a} TM/K_{n+1-a} \arrow[dl, bend left] \\
M &
\end{tikzcd}
$$

Idea relating to brackets

Graded Dirac structures $\sim$ **infinitesimal** version of brackets

(i) S^a is characterized as the subbundle such that $\alpha \in \Omega_H^{a-1}(M)$ if and only if $\mathrm{d}\alpha \in S^a$.

(ii) $\sharp^a$ is characterized as $\{\alpha, \beta\} = (-1)^{\deg_H \beta} \iota_{\sharp_a(\mathrm{d}\alpha)} \mathrm{d}\beta$.

(iii) Involutivity is characterized by the Jacobi identity.

In fact,

Theorem
$\{$*Graded Poisson brackets*$\}^\star \cong \{$*Graded Dirac structures*$\}$

Basic constructions related to graded Dirac structures

Theorem (de León, I.L. 2025)

(i) **Pullbacks**: *$f: M_1 \to M_2$, M_2 is graded Dirac, there is a graded Dirac structure on M_1.*

(ii) **Pushforwards**: *$\pi: M_1 \to M_2$ submersion, M_1 graded Dirac, then* there is a graded Dirac on M_2.*

$\star =$ Vertical vector field define symmetries

Almost regular Lagrangians I

Let $\mathcal{L} = L\,(x^\mu \phi^i \partial_\mu \phi^i)\,\mathrm{d}^n x$. We define the **Legendre transformation** as the fibered derivative:

$$\mathrm{Leg}_{\mathcal{L}} \colon J^1\pi \to \bigwedge^n T^* Y\,.$$

There is a canonical **multisymplectic structure** on $\bigwedge^n T^* Y$, which is a closed, non-degenerate $(n+1)-$form Ω.

We recover the multisymplectic form as $\Omega_{\mathcal{L}} = \mathrm{Leg}^*_{\mathcal{L}}\,\Omega$.

Almost regular Lagrangians II

Generally, working on $(J^1\pi, \Omega_{\mathcal{L}})$ is **hard**, and on $\bigwedge^n T^*Y$ are too many variables. What is easier to work is the submanifold $\operatorname{Im} \mathrm{Leg}_{\mathcal{L}}$ or, rather,the image under:

$$\begin{tikzcd}
J^1\pi \arrow[r, "\mathrm{Leg}_{\mathcal{L}}"] \arrow[dr, "\mathrm{leg}_{\mathcal{L}}"'] & \bigwedge^n T^*Y \arrow[d] \\
& \bigwedge^n T^*Y \big/ \bigwedge^n T^*X
\end{tikzcd}$$

which inherits a natural **graded Dirac structure**.