Coisotropic reduction in Multisymplectic Geometry

Gamma seminar

Rubén Izquierdo, joint work with Manuel de León.

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ICMAT

Outline

1. Symplectic Geometry
2. Multisymplectic Manifolds
3. Hamiltonian multivector fields and forms
4. Coisotropic submanifolds
5. Final remarks and future research

References

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[LDS03] M. de Leon, D. Martín de Diego, and A. Santamaria-Merino. **"Tulczyjew's triples and lagrangian submanifolds in classical field theories".** In: *Applied Differential Geometry and Mechanics*. 2003, pp. 21–47.

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Symplectic Geometry

Definition (Symplectic manifold)

A symplectic manifold is a pair (M, ω), where M is a $2n$-dimensional manifold, and $\omega \in \Omega^2(M)$ is a closed, non-degenerate, 2-form.

Thus, for every symplectic manifold we have an isomorphism induced by contraction

$$TM \xrightarrow{\flat} T^*M; \; v \mapsto \iota_v \omega.$$

Definition

For a subspace $i : W \hookrightarrow T_x M$, define the symplectic orthogonal as

$$W^\perp := \{v \in T_q M, \; \omega(v, w) = 0, \forall w \in W\} = \ker i^* \circ \flat.$$

$$\dim W^\perp = 2n - \dim W$$

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Symplectic orthogonal

Definition

A subspace $W \subseteq T_x M$ (res. submanifold L) is called

- isotropic if $W \subseteq W^{\perp}$ (res. $T_x L \subseteq (T_x L)^{\perp}, \forall x \in L$);
- Lagrangian if $W = W^{\perp}$ (res. $(T_x L)^{\perp} = T_x L, \forall x \in L$).
- coisotropic if $W^{\perp} \subseteq W^{\perp}$ (res. $(T_x L)^{\perp} \subseteq T_x L, \forall x \in L$).

A isotropic submanifold is necessarily n-dimensional and we have the following characterization:

Proposition

An n-dimensional submanifold $i : N \hookrightarrow M$ is Lagrangian if and only if $i^* \omega = 0$.

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Dynamics = Lagrangian submanifolds

Definition

Given a function $H \in C^\infty(M)$ define the Hamiltonian vector field $X_H \in \mathfrak{X}(M)$ as the unique vector field sastifying

$$\iota_{X_H}\omega = dH.$$

A vector field $X \in \mathfrak{X}(M)$ is called locally Hamiltonian if $\iota_X\omega$ is closed.

With the isomorphism $\flat : TM \to T^*M$ we can define

$$\tilde{\omega} := \flat^*\omega_M.$$

Theorem

A vector field $X : M \to TM$ is locally Hamiltonian if and only if it defines a Lagrangian submanifold.

Proof. $X(M)$ is Lagrangian if and only if

$$0 = X^*\tilde{\omega} = -d\iota_X\omega.$$

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Coisotropic reduction

Given a coisotropic submanifold $i : N \hookrightarrow M$, the distribution

$$x \mapsto (T_x N)^\perp$$

is regular and involutive. Therefore, it arises from a maximal foliation $\mathcal{F}$.

Theorem

If $N/\mathcal{F}$ admits a smooth manifold structure such that $\pi : N \to N/\mathcal{F}$ defines a submersion, then there is an unique symplectic form ω_N on $N/\mathcal{F}$ such that

$$\pi^* \omega_N = i^* \omega.$$

Furthermore, if L is a Lagrangian submanifold in M, $\pi(L \cap N)$ is a Lagrangian submanifold in $N/\mathcal{F}$.

Allows for reduction of dynamics!

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Allows for reduction of dynamics!

Proof. We omit the first part. For the projection of Lagrangian submanifolds, let L be a Lagrangian submanifold and denote $L_N := \pi(L \cap N)$.

It is sufficient to see that $\pi(L \cap N)$ is isotropic and that it has maximal dimension in $N/\mathcal{F}$. It is isotropic since $[u] \in T_q(L_N)$ implies $\omega_N([u],[v]) = \omega(u,v) = 0$, for every $[v] \in T_q(L_N)$.Now, since $\ker d_q\pi = (T_qN)^{\perp_\omega}$, the kernel-range formula yields

$$\dim L_N = \dim(L \cap N) - \dim(T_qL \cap (T_qN)^{\perp_\omega}). \tag{1}$$

Furthermore,

$$\dim(L \cap N) + \dim(T_qL + (T_qN)^{\perp_\omega}) = \dim M, \tag{2}$$

because L is Lagrangian and N coisotropic. Substituting (2)in (1) we obtain

$$\begin{aligned}\dim L_N &= \dim M - \dim(T_qL + (T_qN)^{\perp_\omega}) - \dim(T_qL \cap (T_qN)^{\perp_\omega}) \\ &= \dim M - \dim L - \dim(T_qN)^{\perp_\omega} = \dim M - \dim L - (\dim M - \dim N) \\ &= \dim N - \dim L = \dim N - \frac{1}{2}\dim M,\end{aligned}$$

which is exactly $\frac{1}{2}\dim N/\mathcal{F}$, as a direct calculation shows. □

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Definition

For $f, g \in C^\infty(M)$, their Poisson bracket is defined as

$$\{f, g\} := \omega(X_f, X_g).$$

We have the following characterization, which is fundamental for the theory of constraints.

Proposition

A submanifold $i : N \to M$ is coisotropic if and only if, for every pair of functions, f, g constant on N, $\{f, g\} = 0$ on N.

Results to be generalized

1. Endowing TM with the symplectic structure obtained from $\flat : TM \to T^*M$, we can interpret dynamics as Lagrangian submanifolds.
2. Coisotropic submanifolds can be reduced to a symplectic manifold.
3. Lagrangian submanifolds project onto Lagrangian submanifolds under this reduction
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Multisymplectic Manifolds

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Definition

A multisymplectic manifold of order k is a pair (M, ω), where M is a smooth manifold, and ω is a closed $(k+1)-$form.

No non-degeneracy required.

Now we have a collection of maps

$$\bigvee_q M \xrightarrow{\flat_q} \bigwedge^{k+1-q} M; \; U \mapsto \iota_U \omega$$

which endow $\bigvee_q M$ with a multisymplectic structure

$$\widetilde{\Omega}^q_M := \flat_q^* \Omega_M^{k+1-q},$$

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$$W^{\perp,j} := \{v \in T_xM : \iota_{v\wedge w_1\wedge\cdots w_j}\omega = 0, \ \forall w_1, \ldots, w_j \in W\}.$$

Definition

We will say that a subspace $W \subseteq T_xM$ is

- j-isotropic, if
$$W \subseteq W^{\perp,j};$$
- j-coisotropic, if
$$W^{\perp,j} \subseteq W + \ker \flat_1;$$
- j-Lagrangian, if
$$W^{\perp,j} = W + \ker \flat_1.$$

These definitions extend in the natural way to submanifolds.

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Form bundle

Fix a manifold L and define

$$M := \bigwedge^k L.$$

If we define the tautological k-form

$$\Theta_L|_\alpha(v_1, \dots, v_k) = \alpha(\pi_* v_1, \dots, \pi_* v_k),$$

then

$$\Omega_L := d\Theta_L$$

defines a canonical non-degenerate multisymplectic structure on $\bigwedge^k L$. In canonical coordinates $(x, p_{i_1,\dots,i_k})$ representing the form

$$\alpha = p_{i_1,\dots,i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

the canonical multisymplectic form reads

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Lemma

Let (V, ω) be a k-multisymplectic manifold and U, W be k-isotropic and 1-isotropic subspaces respectivley such that

$$V = U \oplus W.$$

Then, U is k-Lagrangian.

Proof. Let $u + w \in U^{\perp,k}$, for $u \in U$, $w \in W$. Then, for all $u_1, \dots, u_k \in U$ we have

$$\omega(u + w, u_1, \dots, u_k) = \omega(w, u_1, \dots, u_k) = 0.$$

We claim that $w \in \ker \flat_1$. Indeed, given $u_i + w_i \in V$,

$$\omega(w, v_1, \dots, v_k) = \omega(w, u_1 + w_1, \dots, u_k + w_k) = \omega(w, u_1, \dots, u_k) = 0,$$

where in the last equality we used that W is 1-isotropic. Therefore, if $u + w \in U^{\perp,k}$, we have

$$u + w \in U + \ker \flat_1,$$

that is

$$U^{\perp,k} \subseteq U + \ker \flat_1,$$

proving that U is k-coisotropic and, therefore, k-Lagrangian □

An useful lemma

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Proposition

A differential k-form

$$\alpha : L \to \bigwedge^k L$$

defines a k-Lagrangian submanifold if and only if it is closed.

Proof. Since the vertical distribution of $L \to \bigwedge^k L$, W, defines a 1-Lagrangian distribution, and $\alpha(L)$ is always complementary to W, it is enough to show that α is k-isotropic. We have

$$\alpha^* \Omega_L = d\alpha,$$

which ends the proof. □

Hamiltonian multivector fields and forms

Definition

Let (M,ω) be a multisymplectic manifold of order k. A multivector field

$$U : M \to \bigvee_q M$$

is called Hamiltonian if there exists a $(k-q)$-form

$$\alpha : M \to \bigwedge^{k-q} M$$

such that

$$\iota_U \omega = d\alpha.$$

We refer to α as the Hamiltonian form. When $\iota_U \omega$ is closed, we call U locally Hamiltonian.

We will denote by the quotient of all Hamiltonian forms ($\Omega_H(M)$) by the space of all closed forms ($Z(M)$)

$$\widehat{\Omega}_H(M) := \Omega_H(M)/Z(M).$$

Defining

$$\deg[\alpha] := k - 1 - (\text{order of } \alpha),$$

and

$$\{[\alpha], [\beta]\}^{\bullet} = -(1)^{\deg \alpha + 1}[\iota_{U \wedge V}\omega],$$

where

$$\iota_U \omega = d\alpha, \iota_V \omega = d\beta,$$

we have

Theorem

For every multisymplectic manifold, $(\widehat{\Omega}_H(M), \{\cdot, \cdot\}^{\bullet})$ *is a graded Lie algebra.*

In particular,

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Bracket of Hamiltonian forms

Let us restrict our attention to currents ($(k-1)$-forms). Defining (without quatienting)

$$\{\alpha, \beta\} = \iota_{U \wedge V}\omega,$$

$\{\cdot, \cdot\}$ does not satisfy the Jacobi identity. Nevertheless,

$$\{\alpha, \{\beta, \gamma\}\} + \text{cycl.} = d\iota_{U \wedge V \wedge W}\omega.$$

This gives an L_∞–algebra structure on

$$\Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-2}(M) \xrightarrow{d} \Omega^{k-1}_H(M).$$

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Theorem ([LI24])

A mutivector field $U : M \to \bigvee_q M$ is locally Hamiltonian if and only if it defines a $(k+1-q)-$Lagrangian submanifold.

Proof. Since $U(M)$ defines a $(k+1-q)-$isotropic submanifold, it follows from the decomposition

$$T\bigvee_q M\bigg|_{U(M)} = TU(M) \oplus \widetilde{W}^{k+1-q},$$

where $\widetilde{W}^{k+1-q}$ is 1-isotropic. □

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Coisotropic submanifolds

Coisotropic submanifolds and brackets

Proposition

If $i : N \hookrightarrow M$ is a k-coisotropic submanifold, the subspace of $(k-1)$-forms wich are closed on N,

$$I_N = \{[\alpha] \in \Omega^{k-1}_H(M) : \ d\alpha = 0 \ \text{on } N\}$$

defines a subalgebra of the Lie algebra $\widehat{\Omega}^{k-1}_H(M)$.

Proof. Let $\hat{\alpha}, \hat{\beta} \in \hat{I}_N$. Then, there are vector fields X_α, X_β satisfying

$$\iota_{X_\alpha}\omega = d\alpha, \ \iota_{X_\beta}\omega = d\beta.$$

Since $i^*d\alpha, i^*d\beta = 0$, we conclude that X_α, X_β take values in $(TN)^{\perp,k} \subseteq TN + \ker \flat_1$. Without loss of generality, we can assume that X_α, X_β take values in TN. Now, since

$$\{\hat{\alpha}, \hat{\beta}\}^\bullet = (-1)^{(k-1)}\iota_{\widehat{X_\alpha \wedge X_\beta}}\omega, i^*\left(\iota_{X_\alpha \wedge X_\beta}\omega\right) = 0,$$

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Thus, when it is regular, there exists a foliation consisting of maximal leaves of the distribution, $\mathcal{F}$. Then,

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Let L be a manifold and $\mathcal{E}$ be a regular distribution on L. Define:

$$\bigwedge_r^k L = \{\alpha \in \bigwedge^k L : \ \iota_{e_1 \wedge \cdots \wedge e_r} \alpha = 0, \forall e_1, \ldots, e_r \in \mathcal{E}\}.$$

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$(\bigwedge_r^k L, \Omega_L)$ is a non-degenerate multisymplectic manifold, where Ω_L is (the restriction of the) canononical multisymplectic form.

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A multisymplectic manifold of type (k, r) is a tuple $(M, \omega, W, \mathcal{E})$, where (M, ω) is a non-degenerate multisymplectic manifold, W is a regular, integrable, 1-Lagrangian distribution, and $\mathcal{E}$ is a subbundle of TM/W satisfying

a) $\iota_{e_1 \wedge \cdots \wedge e_r} \omega = 0$, for all $e_i \in TM$ such that $e_i + W \in \mathcal{E}$;

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An example of coisotropic reduction

Let L be a smooth manifold, $i : Q \subseteq L$ be a submanifold, and $\mathcal{E}$ be a regular distribution. Then,

Proposition

$N := \bigwedge_r^k L\big|_Q$ defines a k-coisotropic submanifold, and for $\alpha \in N$,

$$(T_\alpha N)^{\perp,k} \cong 0 \oplus \ker i^*,$$

where

$$i^* : \bigwedge_r^k L \to \bigwedge_r^k Q$$

is the restriction. Here, the vertical forms are taken with respect to $\tilde{\mathcal{E}} = \mathcal{E} \cap TQ$ (not necessarily of constant rank).

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Proof. ($r = 0$ for symplicity) To prove the previous equality, we just need to prove it in the linear case

$$U \oplus \bigwedge^k L^* \subseteq L \oplus \bigwedge^k L^*.$$

Let $(v, \alpha) \in (U \oplus \bigwedge^k L^*)^{\perp,k}$. Then,

1. $0 = \Omega_L((v,\alpha),(0,\beta),(u_2,0),\dots,(u_k,0)) = \beta(v,u_2,\dots,u_k)$, which implies $v = 0$.
2. $0 = \Omega_L((v,\alpha),(u_1,0),\dots,(u_k,0)) = \alpha(u_1,\dots,u_k)$, which implies

$$\alpha \in \ker i^*.$$

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When $\tilde{\mathcal{E}} = \mathcal{E} \cap Q$ has constant rank, $(TN)^{\perp,k}$ has constant rank and thus, is an involutive distribution. For $x \in Q$, the leaf through $(x, 0) \in N$ is

$$\ker\left(i^* : \bigwedge_r^k T_x^* L \to \bigwedge_r^k T_x^* Q\right).$$

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For $N = \bigwedge_r^k L\big|_Q$, where $TQ \cap \mathcal{E}$ has constant rank,

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Projection of Lagrangian submanifolds (example)

Define a Lagrangian submanifold through a closed form

$$\alpha : L \to \bigwedge_r^k L.$$

Then, the projection to the quotient

$$\pi(\alpha(L) \cap N); \ \pi : N \to N/\mathcal{F}$$

is the image of

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Definition

Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type (k, r). A submanifold $i : N \hookrightarrow M$ is called vertical if $W|_N \subseteq TN$.

Theorem ([LI24])

Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type (k, r), $i : N \hookrightarrow M$ be a vertical k-coisotropic submanifold, and $j : L \hookrightarrow M$ be a k-Lagrangian submanifold complementary to W. Then there is a neighborhood U of L in M, a submanifold $Q \hookrightarrow L$, a neighborhood V of L in $\bigwedge_r^k L$, and a multisymplectomorphism

$$\phi : U \to V$$

satisfying

a) ϕ *is the identity on* L;
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Theorem (Normal form of Lagrangian submanifolds [LDS03])

Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type (k, r), $j : L \hookrightarrow M$ be a k-Lagrangian submanifold complementary to W. Then there is a neighborhood U of L in M, a neighborhood V of L in $\bigwedge^k_r L$, and a multisymplectomorphism

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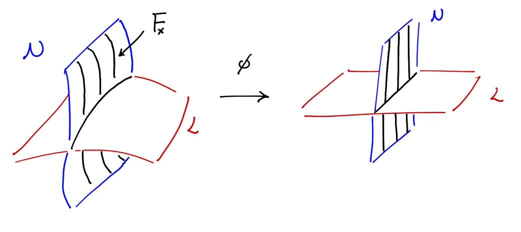
a) ϕ *is the identity on* L;

b) $\phi(U) = V$.

Idea of the proof of local form of coisotropic submanifolds

1. Take
$$\phi : U \to V$$
the multisymplectomorphism of the previous local form, where
$$L \subseteq U \subseteq M;\ L \subseteq V \subseteq \bigwedge^{k}_{r} L.$$
2. Define $Q := \phi(L \cap N)$.
3. Since ϕ preserves the vertical distributions, it also preserves their leaves and then,
$$\phi(U \cap N) = \bigwedge^{k}_{r} L\big|_{Q} \cap V.$$



This local characterization allows us to prove:

Theorem ([LI24])

Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type (k, r), $i : N \hookrightarrow M$ be a vertical k-coisotropic submanifold, and $j : L \hookrightarrow M$ be k-Lagrangian submanifold complementary to W. If $TN/W \cap \mathcal{E}$ has constant rank, so does $(TN)^{\perp,k}$ and we have that, denoting by $\pi : N \to N/\mathcal{F}$ the canonical projection, $\pi(L \cap N)$ is Lagrangian in (N, ω_N).

A general result is not possible, since we can easily find counterexamples.

A counterexample

Let $L = \langle l_1, l_2, l_3 \rangle$ be a 3-dimensional vector space and define

$$V := L \oplus \bigwedge^2 L^*.$$

Let l^1, l^2, l^3 be the dual basis induced on L^* and denote

$$\alpha^{ij} := l^i \wedge l^j.$$

Then

$$V = \langle l_1, l_2, l_3, \alpha^{12}, \alpha^{13}, \alpha^{23} \rangle.$$

Let $l^1, l^2, l^3, \alpha_{12}, \alpha_{13}, \alpha_{23}$ be the dual basis. We have

$$\Omega_L = \alpha_{12} \wedge l^1 \wedge l^2 + \alpha_{13} \wedge l^1 \wedge l^3 + \alpha_{23} \wedge l^2 \wedge l^3.$$

Define

$$N := \langle l_1 + l_2, l_1 + \alpha^{23}, l_2 + \alpha^{13}, l_3, \alpha^{12} \rangle.$$

Then N is a 2-coisotropic subspace. Indeed, a quick calcultion shows $N^{\perp,2} = 0$. This implies that the quotient space $N/N^{\perp,2}$ is (isomorphic to) N. Now, taking as the 2-Lagrangian subspace $L = \langle l_1, l_2, l_3 \rangle$, we have

$$L \cap N = \langle l_1 + l_2, l_3 \rangle.$$

However, this does not define a 2-Lagrangian subspace of $(N, \Omega_L|_N)$, since $\alpha^{12} \in (N \cap L)^{\perp,2}$, but $\alpha^{12} \notin (L \cap W)$.

Final remarks and future research

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- We gave an interpretation of dynamics as Lagrangian submanifolds.
- We proved a coisotropic reduction theorem in a particular class of multisymplectic manifolds.
- For future research we have proposed the following:
  - Apply the results obtained to Field Theories (regularization, constraint analysis, etc)
  - Extend these results to multicontact geometry for the study of dissipative fields.

Thank you for your attention!

Questions?