Recent Developments in the Theory of Brackets in Classical Field Theories

Geometry and Topology for the future III

Rubén Izquierdo-López, joint work with M. de León

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ICMAT-UAM

Structure of the talk

Summary of multisymplectic field theory

Introduction to the problem

Geometric stage and algebraic structure of observables

Hamiltonians and extensions of brackets

References

Summary of multisymplectic field theory

The geometric setting I

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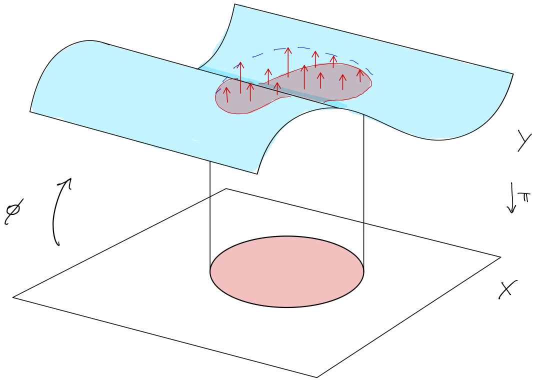
- Fields, denoted by ϕ, are sections of a fibered manifold $Y \xrightarrow{\pi} X$.
- A first order variational problem is defined through a Lagrangian density $\mathcal{L}$ on $J^1\pi_{YX}$ (which defines an n-form on X at each point), with local expression

$$\mathcal{L} = L(x^\mu, y^i, z^i_\mu)\mathrm{d}^n x,$$

and the action can be expressed as

$$\mathcal{J}[\phi] = \int_X (j^1\phi)^* \mathcal{L} = \int_X L(x^\mu, \phi^i, \frac{\partial \phi^i}{\partial x^\mu})\mathrm{d}^n x\,.$$

The geometric setting II



Stationary sections

- We define the multisymplectic form as

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}},$$

where

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x_\mu - \left(z^i_\mu \frac{\partial L}{\partial z^i_\mu} - L \right) d^n x$$

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- Stationary fields (solutions to the field equations) are characterized by

$$(j^1\phi)^* \iota_\eta \Omega_{\mathcal{L}} = 0, \text{ for every } \eta \in \mathfrak{X}(J^1\pi).$$

In coordinates,

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial L}{\partial z^i_\mu}\right) = \frac{\partial L}{\partial y^i}.$$

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Theorem
Let $\omega \in \Omega^{n+1}(M)$ *be a closed form. Then, there is a subfamily of forms* $\Omega^a_H(M) \subseteq \Omega^a(M)$, *for* $0 \leq a \leq n-1$ *and a bracket*

$$\Omega^a_H(M) \otimes \Omega^b_H(M) \xrightarrow{\{\cdot,\cdot\}} \Omega_H^{a+b-(n-1)}(M)$$

that is graded-skew-symemtric and satisfies the graded Jacobi identity up to an exact term.

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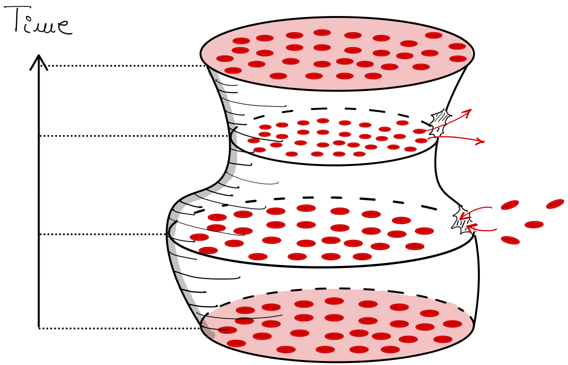
Lagrangian $\rightarrow$ Multisymplectic form $\rightarrow$ Brackets

Introduction to the problem

Conserved quantities in field theories I



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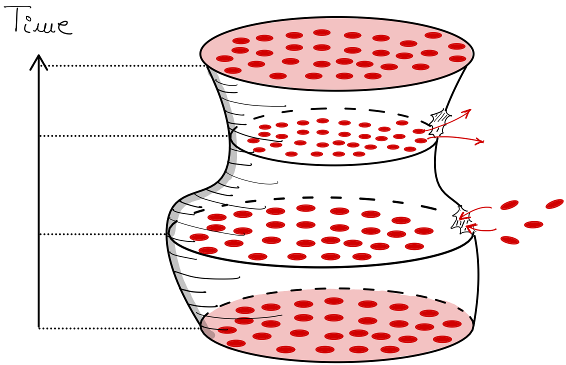
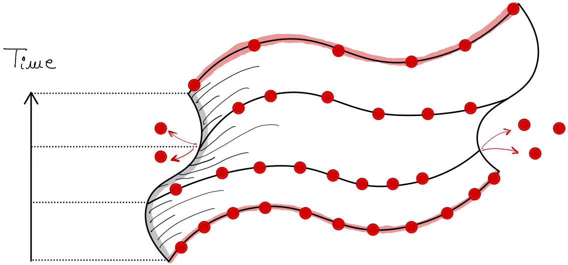


For $\alpha \in \Omega^{n-1}(M)$:

$$\int_{X_{t_1}} \alpha = \int_{X_{t_2}} \alpha - \int_{\partial X \times [t_1, t_2]} \alpha \iff \mathrm{d}\alpha = 0.$$

Conserved quantities in field theories II



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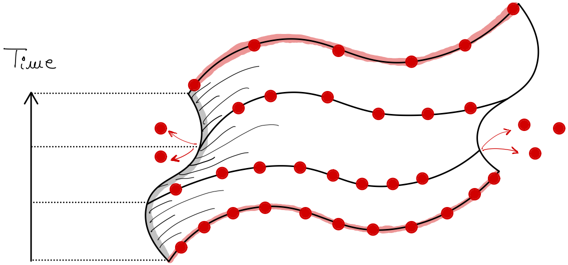

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Problem (that we would like to solve): Find all conserved quantities $\sim$ Find forms that are closed on solutions.

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Problem (that we would like to solve): Find all conserved quantities $\sim$ Find forms that are closed on solutions.

Problem (that we solve): Determine evolution of forms via some bracket:

$$\psi^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}\,,$$

where ψ is a solution of the corresponding PDE and α is some *a*-form.

Some advances:

1. Igor V. Kanatchikov. **"Canonical Structure of Classical Field Theory in the Polymomentum Phase Space".** In: *Rep. Math. Phys.* **41**.1 (1998), pp. 49–90
2. Miguel Á. Berbel and Marco Castrillón López. **"Poisson–Poincaré Reduction for Field Theories".** In: *J. Geom. Phys.* **191** (2023), p. 104879
3. François Gay-Balmaz, Juan C. Marrero, and Nicolás Martínez-Alba. **"A New Canonical Affine Bracket Formulation of Hamiltonian Classical Field Theories of First Order".** In: *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **118**.3 (2024), p. 103

Geometric stage and algebraic structure of observables

General setup for field equations

Let $\tau : M \longrightarrow X$ denote a fibered manifold, $n = \dim X$. Let:

(i) $\alpha_1, \ldots, \alpha_k \in \Omega^{n-1}(M)$ be semi-basic forms (representing observables).

(ii) $\beta_1, \ldots, \beta_k \in \Omega^n(M)$ be semi-basic forms (representing evolution of observables).

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We deal with partial differential equations with the following structure*:

$$\psi^*(\mathrm{d}\alpha_i) = \beta_i \circ \psi \,, \text{where } \psi : X \to M \text{ is a section.}$$

Examples (equations)

(i) Hamilton equations:

(a) Fiber bundle: $T^*Q \times \mathbb{R} \to \mathbb{R}$.

(b) Semi-basic forms: q^i, p_i, t.

(c) Basic forms: $\frac{\partial H}{\partial p_i}\mathrm{d}t\,, -\frac{\partial H}{\partial q^i}\mathrm{d}t$, $\mathrm{d}t$.

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(ii) Hamilton–De Donder–Weyl equations:

(a) Fiber bundle: (covariant phase space) $\bigwedge_2^n Y / \bigwedge_1^n Y \to X$.

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(iii) Yang–Mills equations:

(a) Fiber bundle: $\operatorname{Im}\operatorname{leg}_{\mathcal{L}} \to X$.

(b) Semi-basic forms:

$A^i_\mu \mathrm{d}^{n-1}x_\nu - A^i_\nu \mathrm{d}^{n-1}x_\mu$, $F_i^{\mu\nu}\mathrm{d}^{n-1}x_\nu$, $\frac{1}{n}x^\mu \mathrm{d}^{n-1}x_\mu$.

(c) Basic forms:

$\left(F^i_{\mu\nu} - f^i_{jk}A^j_\nu A^k_\mu\right)\mathrm{d}^n x$, $\left(-f^i_{jk}F_i^{\mu\nu}A^k_\mu\right)\mathrm{d}^n x$, $\mathrm{d}^n x$.

Algebraic structure of observables

The observables $\alpha_1 \ldots, \alpha_k \in \Omega^{n-1}(M)$ allow us to define the space of Hamiltonian forms:

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Assumption 1: There is a bracket $\{\cdot, \cdot\}$ on the space of Hamiltonian forms satisfying the following properties:

(i) It is *skew-symmetric*: $\{\alpha, \beta\} = -\{\beta, \alpha\}$.

(ii) It satisfies the Jacobi identity up to an exact term:

$$\{\alpha, \{\beta, \gamma\}\} + \{\beta, \{\gamma, \alpha\}\} + \{\gamma, \{\alpha, \beta\}\} = \text{exact form} .$$

(iii) It vanishes on closed forms: $\mathrm{d}\alpha = 0 \implies \{\alpha, \beta\} = 0$.

(iv) There is a correspondence $\alpha \mapsto X_\alpha$ such that $\{\alpha, \beta\} = \iota_{X_\beta} \mathrm{d}\alpha$.

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(iii) Yang–Mills equations: Non vanishing brackets:

$$\{A^i_\mu \mathrm{d}^{n-1}x_\nu - A^i_\nu \mathrm{d}^{n-1}x_\mu, F^{\alpha\beta}_j \mathrm{d}^{n-1}x_\beta\} = \delta^i_j \delta^{\alpha\beta}_{\mu\nu} \mathrm{d}^{n-1}x_\beta \,.$$

Hamiltonian forms of arbitrary order I

Definition
We say that a form $\alpha \in \Omega^a(M)$ is special Hamiltonian if there is a semi-basic form $\beta \in \Omega^{a+1}(M)$ such that $\psi^*(\mathrm{d}\alpha) = \beta \circ \psi$, for every solution of the equations. The space of special Hamiltonian a-forms is denoted by $\widetilde{\Omega}^a_H(M)$.

Remark
If $\alpha \in \Omega^{n-1}(M)$ is special Hamiltonian, $\alpha \in \Omega^{n-1}_H(M)$.

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Theorem (de León, Izquierdo-López 2025)
$\alpha \in \Omega^a(M)$ is special Hamiltonian if and only if $\alpha \wedge \varepsilon \in \Omega^{n-1}_H(M)$, for every closed and basic $\varepsilon \in \Omega^{n-1-a}(M)$.

Hamiltonian forms of arbitrary order II

Definition
A form $\alpha \in \Omega^a_H(M)$ is called Hamiltonian if $\mathrm{d}\alpha \in \iota_{\bigwedge^{n-(a+1)} TM}\langle \mathrm{d}\alpha_1, \ldots, \mathrm{d}\alpha_k \rangle$. We denote by $\Omega^a_H(M)$ the space of Hamiltonian a-forms.

Proposition
$\widetilde{\Omega}^a_H(M) \subseteq \Omega^a_H(M)$

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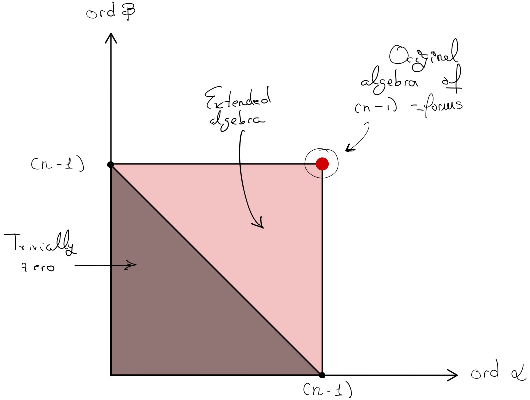
Theorem (de León, Izquierdo-López 2025)
There is an unique induced graded Poisson bracket

$$\Omega^a_H(M) \otimes \Omega^b_H(M) \to \Omega_H^{a+b-(n-1)}(M)$$

that mantains the properties of the original bracket of $(n-1)$*-forms. Furthermore*, special Hamiltonian forms define a subalgebra.*

Summary



ord β
Original algebra of (n−1)-forms
Extended algebra
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Trivially zero
ord α
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Hamiltonians and extensions of brackets

Domain of definition of current brackets

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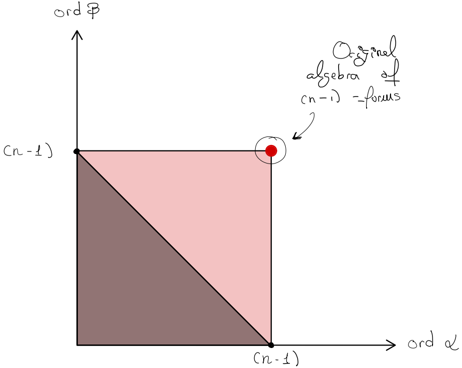

First extension of the brackets I

Theorem (de León-Izquierdo López 2025)
There exists an unique extension of $\{\cdot,\cdot\}$

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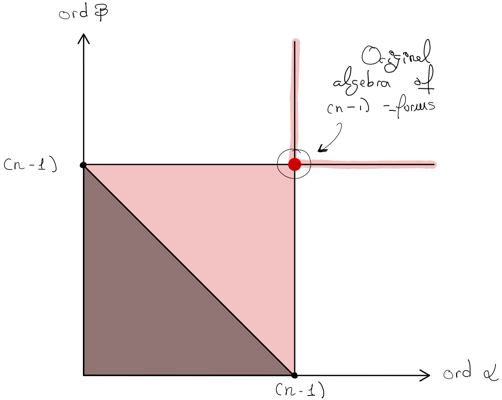
Assumption 2: There exists a form $\mathcal{H} \in \Omega^n(M)[1]$, the Hamiltonian, such that

$$\psi^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}\,,$$

for every solution ψ and $\alpha \in \Omega^{n-1}_H(M)$.

First extension of the brackets II

Current domain of definition:



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Hamiltonian = Poincaré–Cartan form

Final extension of the bracket I

Question: Can we interpret $\psi^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}$, for arbitrary $\alpha \in \Omega^a_H(M)$?

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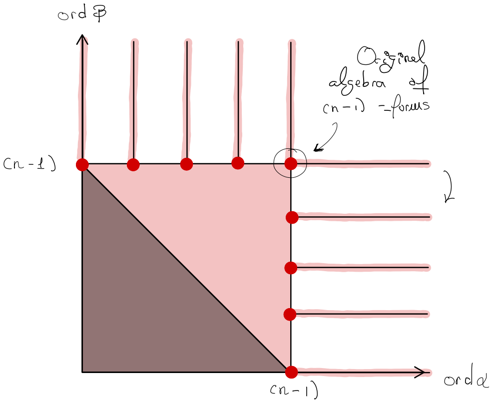

Final extensions of the brackets II

Problem: There is no unique extension of the bracket to order $a < n - 1$.

Nevertheless,

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Nevertheless,

Theorem (de León, Izquierdo-López 2025)
There is a bijective correspondence between the possible extensions of the bracket and affine maps

$$\gamma : \{\textit{Hamiltonians}\} \to \{\textit{Ehresmann connections on } \tau : M \to X\}$$

such that $\gamma(\mathcal{H})$ solves the Hamilton–De Donder–Weyl equations of $\mathcal{H}$, for every $\mathcal{H}$.

Final extension of the bracket III

Corollary
Let γ be such a map, $\mathcal{H}$ be a Hamiltonian, and ψ be an integral section of $\gamma(\mathcal{H})$. Then

$$\psi^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}_\gamma \,,$$

for every $\alpha \in \Omega^a_H(M)$.

Final extension of the bracket III

Corollary
Let γ be such a map, $\mathcal{H}$ be a Hamiltonian, and ψ be an integral section of $\gamma(\mathcal{H})$. Then

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for every $\alpha \in \Omega^a_H(M)$.

Corollary
Let α be a special Hamiltonian form. Then, the bracket $\{\alpha, \mathcal{H}\}_\gamma$ is independent of extension $\{\cdot, \cdot\}_\gamma$, for every Hamiltonian $\mathcal{H}$.

Technical remarks

The construction was based on a generalization of the $\sharp$ mapping associated to a graded Poisson bracket. In particular, we generalized the techiniques employed in

1. Peter W. Michor. **"A Generalization of Hamiltonian Mechanics".** In: *J. Geom. Phys.* **2**.2 (1985), pp. 67–82
2. Janusz Grabowski. **"Z-Graded Extensions of Poisson Brackets".** In: *Rev. Math. Phys.* **09**.01 (1997), pp. 1–27

to extend the brackets.

Final remarks and remaining questions

(i) The previous theoretical results seem to indicate that the subalgebra of special Hamiltonian forms is of high relevance to a particular field theory. We would like to compute these subelgebras for several almost regular Lagrangians to further study these classical field theories.

(ii) We would also like to investigate the relation between these extensions and the instantanous split formalism.

(iii) It is also interesting to investigate the implications of this algebraic structure in the study of momentum maps and reduction, employing the graded brackets.

References

Main references

1. Manuel de León and Rubén Izquierdo-López. **“Graded Poisson and Graded Dirac Structures”.** In: *J. Math. Phys.* **66**.2 (2025). 10.1063/5.0243128, p. 022901
2. Manuel de León and Rubén Izquierdo-López. ***A description of classical field equations using extensions of graded Poisson brackets*****.** *Soon in arXiv.* 2025

Thank you for your attention!