Coisotropic reduction in Symplectic, Cosymplectic, Contact, and Cocontact geometry

BYMAT 2023

Rubén Izquierdo López

Universidad Complutense de Madrid-ICMAT

12 de noviembre de 2023

The symplectic case

Definition 0.1 (Symplectic manifold)

A symplectic manifold is a manifold pair (M, ω)*, where* M *is a manifold, and* ω *is a closed, non-degenerate* 2*-form; where non-degeneracy means that*

$$v \mapsto \iota_v \omega$$

is a diffeomorphism between TM *and* T^*M*.*

Definition 0.2 (Symplectic orthogonal, coisotropic and Lagrangian submanifold)

Definition 0.2 (Symplectic orthogonal, coisotropic and Lagrangian submanifold)

Given a subspace $W \subset T_q M$, *for some* $q \in M$, *we define the symplectic orthogonal*

$$W^{\perp} := \{v \in T_q M \,|\, \omega(v, w) = 0, \forall w \in W\}.$$

We say that a submanifold $i : N \hookrightarrow M$ *is coisotropic if*

$$T_q N^{\perp} \subset T_q N, \forall q \in N,$$

and say that it is Lagrangian if

$$T_q N^{\perp} = T_q N, \forall q \in N.$$

Definition 0.2 (Symplectic orthogonal, coisotropic and Lagrangian submanifold)

Given a subspace $W \subset T_q M$*, for some* $q \in M$*, we define the symplectic orthogonal*

$$W^\perp := \{v \in T_q M \,|\, \omega(v, w) = 0, \forall w \in W\}.$$

We say that a submanifold $i : N \hookrightarrow M$ *is coisotropic if*

$$T_q N^\perp \subset T_q N, \forall q \in N,$$

and say that it is Lagrangian if

$$T_q N^\perp = T_q N, \forall q \in N.$$

On every coisotropic submanifold N, we can define the coisotropic distribution $TN^\perp$(smooth selection of subspace of the tangent space at every point):

$$q \mapsto T_q N^\perp \subset T_q N.$$

Proposition 0.1

The coisotropic distribution $TN^{\perp}$ is integrable, that is, arises from a maximal foliation $\mathcal{F}$ of N.

Proposition 0.1

The coisotropic distribution $TN^{\perp}$ is integrable, that is, arises from a maximal foliation $\mathcal{F}$ of N.

Theorem 1 (Weinstein coisotropic reduction theorem)

If $N/\mathcal{F}$ has a manifold structure such that the canonical projection $\pi : N \to N/\mathcal{F}$ defines a submersion, there exists an unique symplectic form ω_N on $N/\mathcal{F}$ such that

$$\pi^*\omega_N = i^*\omega.$$

Furthermore, let $L \hookrightarrow M$ be a Lagrangian submanifold that has clean intersection with N ($T_qN \cap L = T_qN \cap T_qL$). Then, if $\pi(L \cap N)$ is a submanifold, it is Lagrangian in $(N/\mathcal{F}, \omega_N)$.

The Cosymplectic, Contact, and Cocontact case.

Definition 0.3 (Cosymplectic, Contact and Cocontact manifolds)

- *A cosymplectic manifold is a triple* (M, ω, θ), *where* M *is a* $(2n+1)$*-dimensional manifold,* ω *and* θ *are closed* 2 *and* 1 *forms, respectively, such that*

$$v \mapsto \iota_v \omega + \theta(v)\theta$$

is a diffeomorphism.

The Cosymplectic, Contact, and Cocontact case.

Definition 0.3 (Cosymplectic, Contact and Cocontact manifolds)

- *A cosymplectic manifold is a triple* (M, ω, θ), *where* M *is a* $(2n+1)$*-dimensional manifold,* ω *and* θ *are closed* 2 *and* 1 *forms, respectively, such that*

$$v \mapsto \iota_v \omega + \theta(v)\theta$$

is a diffeomorphism.

- *A contact manifold is a couple* (M, η), *where* M *is a* $(2n+1)$*-dimensional manifold, and* η *is a* 1*-form such that*

$$v \mapsto \iota_v d\eta + \eta(v)\eta$$

is a diffeomorphism.

Definition 0.4 (Cosymplectic, Contact and Cocontact manifolds)

A cocontact manifold is a triple (M, η, θ)*, where* M *is a* $(2n+2)$*-dimensional manifold,* η *and* θ *are* 1*-forms (*θ *closed) such that*

$$v \mapsto \iota_v d\eta + \eta(v)\eta + \theta(v)\theta$$

is a diffeomorphism.

The previous diffeomorphisms will be denoted by $\flat$, and $\sharp := \flat^{-1}$.

Definition 0.4 (Cosymplectic, Contact and Cocontact manifolds)

A cocontact manifold is a triple (M, η, θ), where M is a $(2n+2)$-dimensional manifold, η and θ are 1-forms (θ closed) such that

$$v \mapsto \iota_v d\eta + \eta(v)\eta + \theta(v)\theta$$

is a diffeomorphism.

The previous diffeomorphisms will be denoted by $\flat$, and $\sharp := \flat^{-1}$. This allows us to define the bivector field:

$$\Lambda_q(\alpha_q, \beta_q) := \begin{cases} \omega_q(\sharp\alpha_q, \sharp\beta_q) \text{ in the cosympelctic case,} \\ -d\eta(\sharp\alpha_q, \sharp\beta_q), \text{ in the contact and cocontact case} \end{cases}$$

The bivector Λ defines a canonical morphism

$$\sharp_{\Lambda} : T^*M \to TM;\ \alpha_q \mapsto \iota_{\alpha_q}\Lambda_q.$$

The bivector Λ defines a canonical morphism

$$\sharp_\Lambda : T^*M \to TM; \; \alpha_q \mapsto \iota_{\alpha_q}\Lambda_q.$$

Definition 0.5 (Λ-orthogonal, coisotropic)

Given a subspace $W \subset T_qM$, we define

$$W^{\perp_\Lambda} := \sharp_\Lambda(W^0),$$

where $W^0 \subset T_q^*M$ is the annhilator of W. We say that a submanifold $i : N \to M$ is coisotropic if

$$T_qN^{\perp_\Lambda} \subset T_qN, \forall q \in N.$$

Furthermore, we have the natural distributions:

- In cosymplectic manifolds:

$$\mathcal{H} := \operatorname{Ker}\theta, \mathcal{V} := \operatorname{Ker}\omega.$$

Furthermore, we have the natural distributions:

- In cosymplectic manifolds:

$$\mathcal{H} := \operatorname{Ker}\theta, \mathcal{V} := \operatorname{Ker}\omega.$$

- In contact manifolds:

$$\mathcal{H} := \operatorname{Ker}\eta, \mathcal{V} = \operatorname{Ker}d\eta.$$

Furthermore, we have the natural distributions:

- In cosymplectic manifolds:

$$\mathcal{H} := \operatorname{Ker}\theta, \mathcal{V} := \operatorname{Ker}\omega.$$

- In contact manifolds:

$$\mathcal{H} := \operatorname{Ker}\eta, \mathcal{V} = \operatorname{Ker} d\eta.$$

- In cocontact manifolds:

$$\mathcal{H} = \operatorname{Ker}\eta \cap \operatorname{Ker}\theta,$$

$$\mathcal{V}_z = \operatorname{Ker} d\eta \cap \operatorname{Ker}\theta, \mathcal{H}_z = \operatorname{Ker}\theta$$

$$\mathcal{V}_t = \operatorname{Ker} d\eta \cap \operatorname{Ker}\eta, \mathcal{H}_t = \operatorname{Ker}\eta$$

Symplectic	Arbitrary	$(N/\mathcal{F}, \omega_N)$, symplectic
Cosymplectic	Vertical	$(N/\mathcal{F}, \theta_N, \Omega_N)$, cosymplectic
	Horizontal	(N, Ω_N), symplectic
	Arbitrary	Foliation consisting of symplectic manifolds of $N/\mathcal{F}$
Contact	Vertical	$(N/\mathcal{F}, \eta_N)$, contact
	Horizontal	$\dim N/\mathcal{F} = 0$
Cocontact	tz-vertical	$(N/\mathcal{F}, \theta_N, \eta_N)$, cocontact
	t-vertical, z-horizontal	$\dim N/\mathcal{F} = 1$, $\theta_N \neq 0$
	z-vertical, t-horizontal	$(N/\mathcal{F}, \eta_N)$, contact
	tz-horizontal	$\dim N/\mathcal{F} = 0$
SHS	Vertical	$(N/\mathcal{F}, \omega_N, \lambda_N)$, stable Hamiltonian
	Horizontal	$(N/\mathcal{F}, \omega_N)$, symplectic