# Coisotropic reduction in Multisymplectic Geometry

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### Resumen

En este trabajo de fin de Máster estudiamos la reducción coisótropa en geometría multisimpléctica. Por un lado, damos una interpretación de los campos de multivectores Hamiltonianos como subvariedades Lagrangianas y demostramos que una subvariedad $k$-coisotrópica induce un subálgebra de Lie en el álgebra de las $(k-1)-$formas Hamiltonianas. Por otro lado, extenemos el resultado clásico de geomertía simpléctica de proyección de subvariedades Lagrangianas en la reducción coisótropa a fibrados de formas, que están dotados de una estructura multisimpléctica natural.



### Abstract

In this text we study coisotropic reduction in multisymplectic geometry. On the one hand, we give an interpretation of Hamiltonian multivector fields as Lagrangian submanifolds and prove that $k$-coisotropic submanifolds induce a Lie subalgebra in the algebra of Hamiltonian $(k-1)$-forms, similar to how coisotropic submanifolds in symplectic geometry induce a Lie subalgebra under the Poisson bracket. On the other hand, we extend the classical result of symplectic geometry of projection of Lagrangian submanifolds in coisotropic reduction to bundles of forms, which naturally carry a multisymplectic structure.



### Keywords

Multisymplectic manifolds, coisotropic submanifolds, Lagrangian submanifolds, coisotropic reduction, graded Lie algebras.

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# Introduction

Multisymplectic geometry is the natural framework in which to formulate classical field theories, just as symplectic geometry plays that key role in Lagrangian and Hamiltonian mechanics [AM08; BSF88; LR89]. Indeed, the bundles of exterior forms are naturally equipped with a multisymplectic form, in the same way that for the bundle of 1-forms (i.e. the cotangent bundle of the manifold) the natural structure is a symplectic form. However, multisymplectic geometry exhibits a much higher degree of complexity, dealing with differential forms of higher degree. These differences make multisymplectic geometry richer but at the same time more complicated, and if the holy grail of classical field theories is to seek a full extension of the results in symplectic mechanics, this task is far from being fully achieved. This paper tries to cover some aspects that have already been partially dealt with in previous papers [CIL99; CIL96], thus initiating an ambitious plan that we hope to complete in the coming years.

One of the key aspects of this new approach is not to consider any notion of regularity in the definition of a multisymplectic form, as is usually done in applications to classical field theories [Got88; Got91a; Got91b; CIL99; Rom09a; Rom09b; RW19]. This allows us to work with greater flexibility, recovering regularity as a particular case. Our main objective in this text is to study the submanifolds of a multisymplectic manifold, in particular the relations between Lagrangian and coisotropic submanifolds [CIL99; LDS03; SW19]. In doing so, we prove a coisotropic reduction theorem which generalises the one already known for symplectic geometry. The interest of this reduction lies in the fact that the Lagrangian submanifolds are the geometric interpretation of the dynamics, and if one of them has a clean intersection with a coisotropic one, it can be reduced to the quotient of the latter while maintaining the Lagrangian character (and so, providing a reduced dynamics) [Wei71; AM08]. Very relevant by-products of these notions and results are the construction of graded brackets and the interpretation of a coisotropic submanifold in terms of these brackets, as well as the study of currents and conserved quantities [CIL96; FPR03; Bla21] (see also [LMS04; RWZ20]).

This master thesis is structured as follows. In Chapter 1 we motivate the study of multisymplectic manifolds (the main object of study in this text) through Calculus of Variations. In the last section of this chapter we study the particular case of Classical Mechanics, and present the main results that we aim to generalize. Then, in Chapter 2 we develop an abstract theory and introduce the main concepts that are going to be employed in the last chapters. The main results of this master thesis can be found in Chapter 3 and in Chapter 4, where we give the interpretation of Hamiltonian multivector fields as Lagrangian submanifolds, and prove a theorem of coisotropic reduction, respectively. Finally, we have included two appendices, explaining in more detail some aspects of Symplectic geometry (Appendix A) and the first Jet Bundle (Appendix B).

# Chapter 1

# Classical Field Theory (Calculus of Variations)

This Chapter introduces the use of Multisymplectic Geometry in Classical Field Theory (or in Calculus of Variations). Although not strictly necessary to understand Chapter 2, Chapter 3 or Chapter 4, we believe that with a brief introduction to the origin of Multisymplectic Geometry one can grasp better the motivation for the results obtained in this text. Since Chapter 1 has a very different objective than the rest of this master thesis, the reader will find that the exposition differs fundamently from the other chapters, focusing more on the motivation rather than on details and rigor (which is the focus in the latter chapters). The structure presented is heavily inspired from [BSF88] which, as a side note, has great historial remarks. However, a rather unusual choice has been made for the exposition, to study Classical Mechanics *after* Classical Field Theory. This is not arbitrary, the objective being to have a section where we present the main results we aim to generalize after the reader has understood the geometric formalism of Calculus of Variations and just before the beginning of the in-depth study of multisymplectic manifolds.

The original objective of Multisymplectic Geometry (although not with the modern name) was to obtain a way of expressing the Euler-Lagrange equations of Calculus of Variations in an intrinsic manner. This idea goes back to Cartan in his work on Classical Mechanics [Car22], and was later employed in general Calculus of Variations by himself and De Donder [Car33], [DeD29]. The presentation we give tries to follow this philosophy and, therefore, we begin studying Calculus of variations in Section 1.1, where we introduce the notion of jet bundles (although not in great detail) to have an intrinsic formulation of the variational problem. Since the intrinsic Euler-Lagrange equations require more machinery, we focus on obtaining the local Partial Differential Equations. Obtaining the instrinsic equations is the objective of Section 1.2, where we define the Poincaré-Cartan-form and the multisymplectic form of the theory. The Hamiltonian formalism of Calculus of Variations is obtained via the change of variables $p_i^\mu = \frac{\partial L}{\partial z^i_\mu}$, where $p_i^\mu$ are the conjugate momenta to $z^i_\mu$. This leads to the Hamilton-De Donder-Weyl equations. The previous coordinate transformation can be interpreted geometrically as a diffeomorphism between two multisymplectic manifolds (under certain regularity hypotheses), which is called the Legendre transformation, and is studied in Section 1.3. Finally, we study the particular case of time-independent Classical Mechanics in Section 1.4, where we present the symplectic structure on the tangent bundle and the main results that we are going to generalize in future chapters.

## 1.1 The Lagrangian Formalism

A **field** is a section $\phi : X \to Y$ of some fibered manifold

$$\pi : Y \to X,$$

where $X$ is an oriented $n$-dimensional manifold. Similar to Classical Mechanics, we will look for fields that extremize certain functional $S[\phi]$. Tipically, this functional will be of the form

$$S[\phi] = \int_X \mathcal{L}(\phi),$$

where $\mathcal{L}(\phi)$ is an $n$-form on $X$[1] (called **the Lagrangian density**). In this text, we will deal with *first order* field theories. More precisely, theories where the Lagrangian depends up to the first derivate. In fibered coordinates $(x^\mu, y^i) \xrightarrow{\pi} (x^\mu)$, this means

$$\mathcal{L}(\phi) = L\left(x^\mu, \phi^i, \frac{\partial \phi^i}{\partial x^\mu}\right) d^n x,$$

where we use the notation

$$\phi^i = y^i \circ \phi,$$
$$d^n x = dx^1 \wedge \cdots \wedge dx^n.$$

If $X$ is oriented with a volume form $\eta$, we can canonically define $L_\eta$ as the previous function choosing coordinates such that $d^n x = \eta$. We will later see that the theory does not depend on the choice of $\eta$ (in fact, $X$ may not even be orientable).

More rigourosly, the Lagrangian density can be seen as a morphism of fibered manifolds

$$\mathcal{L} : J^1\pi \to \bigwedge^n X,$$

where $J^1\pi$ denotes the *first jet bundle*, a manifold in which each point represents a section up to its first derivatie (for a more detailed explanation see Appendix B). Each set of fibered coordinates $(x^\mu, y^i)$ induces natural coordinates on $J^1\pi$, $(x^\mu, y^i, z^i_\mu)$, where $z^i_\mu$ represents $\frac{\partial y^i}{\partial x^\mu}$. Then, to each section

$$\phi : X \to Y$$

[1] In the case that $X$ is not compact, the integral may not be well-defined, but we will take care of this incovinience later on.

there is the corresponding **jet lift**

$$j^1\phi : X \to J^1\pi,$$

asigning to each point $x \in X$ the element $j^1_x\phi$ representing $\phi(x)$ and $d_x\phi$. In coordinates, the jet lift reads

$$(j^1\phi)^* x^\mu = x^\mu;$$
$$(j^1\phi)^* y^i = \phi^i;$$
$$(j^i\phi)^* z^i_\mu = \frac{\partial z^i}{\partial x^\mu}.$$

With this formalism, the action is intrinsically defined as

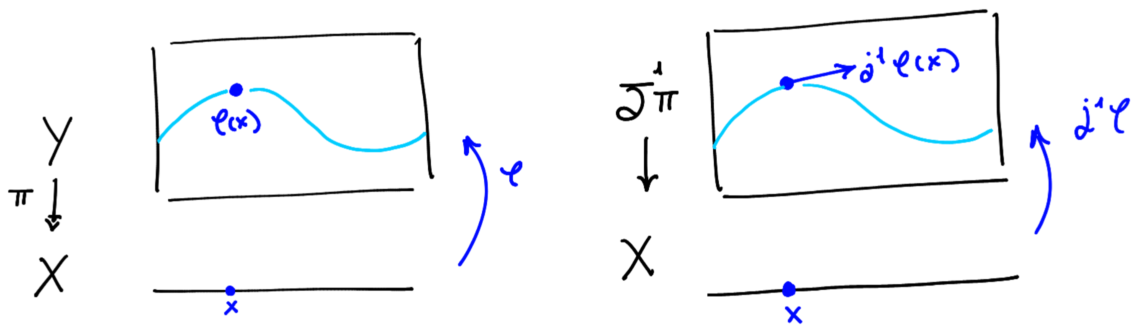



Figure 1.1: Jet lift, $\phi$ represented up to its first derivative

$$S[\phi] = \int_X \mathcal{L}\circ(j^1\phi).$$

As mentioned, this may be ill-defined. To salvage this, we can look for fields $\phi$ that *locally extremize* $S$. More particularly, fields which extremize

$$S_D[\phi] := \int_D \mathcal{L}\circ(j^1\phi),$$

for each $n$-dimensional compact submanifold $D \subseteq X$, and *fixed* values on $\partial D$. We can also write the action depending on a $n$-form on $J^1\pi$ as

$$S_D[\phi] = \int_D (j^1\phi)^* \widetilde{\mathcal{L}},$$

where

$$\widetilde{\mathcal{L}}(j^1\phi) := \tau^* \left(\mathcal{L}(j^1\phi)\right),$$

and

$$\tau : J^1\pi \to X$$

is the natural projection. For the sake of symplicity, we will write $\widetilde{\mathcal{L}} = \mathcal{L}$. Of course, the $n$-form $\mathcal{L}$ has the local expression

$$\mathcal{L} = L(x^\mu, y^i, z^i_\mu)d^n x.$$

When a volume form $\eta$ is fixed on $X$, we can write (identifying $\eta$ with its pull-back to $J^1\pi$)

$$\mathcal{L} = L_\eta \eta.$$

Then, $L_\eta$ is called the **Lagrangian function**.

**Example 1.1.1.** *As an example of a possible first order field theory, we have Classical Mechanics. Indeed, let*

$$Y := Q \times \mathbb{R} \xrightarrow{\pi} \mathbb{R} =: X$$

*be the canonical projection, and* $\eta := dt$ *be the canonical volume form on* $\mathbb{R}$. *Then, the first jet bundle can be canonically identified with*

$$J^1\pi = TQ \times \mathbb{R}.$$

*A Lagrangian density will be a* 1*-form*

$$\mathcal{L} = L(t, q, \dot{q})dt,$$

*and the correspoding variational problem will be the classical Principle of Least Action, with action functional*

$$S_{[t_0,t_1]}[\gamma] = \int_{t_0}^{t_1} L(t, \gamma(t), \dot{\gamma}(t))dt.$$

Let us obtain the field equations through the variational principle. A variation of $\phi$ will be a 1-parameter smooth family of sections $\phi_t$ with $\phi_0 = \phi$, $\phi_t|_{\partial D} = \phi|_{\partial D}$. The extremal condition will read

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} S_D(\phi_t) = 0.$$

It is easy to see that it is enough to look for variations of the form

$$\phi_t = \varphi_t \circ \phi,$$

where $\varphi_t$ is the (possibly local) flow of a vertical vector field $\xi$ satisfying

$$\xi\big|_{\partial D} = 0.$$

We have

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} S_D(\phi_t) = \int_D \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (j^1(\varphi_t \circ \phi))^* \mathcal{L}. \tag{1.1}$$

We can write $j^1(\varphi_t \circ \phi)$ in a more convenient way. First, for each fibered manifold morphism

$$\varphi : Y \to Y,$$

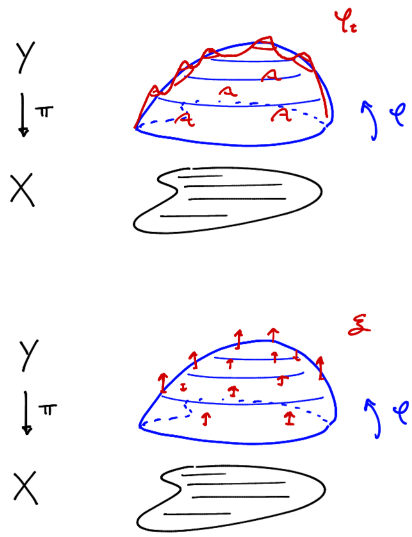



Figure 1.2: A variation and an infinitesimal variation

define the **jet lift** of $\varphi$

$$\varphi^{(1)} : J^1\pi \to J^1\pi$$

as

$$(\varphi^{(1)})(j^1\phi) := j^1(\varphi\circ\phi).$$

Locally,

$$\begin{aligned}
(\varphi^{(1)})^* x^\mu &= x^\mu;\\
(\varphi^{(1)})^* y^i &= \varphi^i;\\
(\varphi^{(1)})^* z^i_\mu &= \frac{\partial \varphi^i}{\partial x^\mu} + \frac{\partial \varphi^i}{\partial y^j} z^j_\mu.
\end{aligned}$$

It is easy to see that we have

$$(\phi\circ\psi)^{(1)} = \phi^{(1)}\circ\psi^{(1)}$$

and thus, $(\phi_t)^{(1)}$ is the flow of certain vector field, which we denote $\xi^{(1)}$ and define to be the **jet lift** of $\xi$. If the local expression of $\xi$ is

$$\xi = \xi^i \frac{\partial}{\partial y^i},$$

then the local expression of $\xi^{(1)}$ is

$$\xi^{(1)} = \xi^i \frac{\partial}{\partial y^i} + \left( \frac{\partial \xi^i}{\partial x^\mu} + \frac{\partial \xi^i}{\partial y^j} z^j_\mu \right) \frac{\partial}{\partial z^i_\mu}.$$

Therefore, the extremal condition can be expressed as

$$0 = \int_D \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (j^1(\varphi_t\circ\phi))^*\mathcal{L} = \int_D \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\varphi_t^{(1)}\circ j^1\phi)^*\mathcal{L} = \int_D (j^1\phi)^* \pounds_{\xi^{(1)}}\mathcal{L},$$

for every vertical vector field $\xi \in \mathfrak{X}(Y)$ vanishing on $\partial D$. However, this is not a satisfactory result since we would hope for *equations* for $\phi$.

Let us perform some calculations in coordinates and obtain the aforementioned field equations to motivate the construction that leads to the intrinsic version. We have

$$£_{\xi^{(1)}}\mathcal{L} = \xi^i \frac{\partial L}{\partial y^i} d^n x + \left( \frac{\partial \xi^i}{\partial x^\mu} + \frac{\partial \xi^i}{\partial y^j} z^j_\mu \right) \frac{\partial L}{\partial z^i_\mu} d^n x,$$

and thus (writing the corresponding evaluations),

$$\begin{aligned}
(j^1\phi)^* £_{\xi^{(1)}}\mathcal{L} &= \xi^i(x^\mu, \phi^i) \frac{\partial L}{\partial y^i} d^n x + \left( \frac{\partial \xi^i}{\partial x^\mu}(x^\mu, \phi^i) + \frac{\partial \xi^i}{\partial y^j}(x^\mu, \phi^i) \frac{\partial \phi^j}{\partial x^\mu} \right) \frac{\partial L}{\partial z^i_\mu} d^n x \\
&= \xi^i(x^\mu, \phi^i) \frac{\partial L}{\partial y^i} d^n x + \frac{\partial}{\partial x^\mu} \left( \xi^i(x^\mu, \phi^i) \right) \frac{\partial L}{\partial z^i_\mu} d^n x \\
&= \xi^i \left( \frac{\partial L}{\partial y^i} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial z^i_\mu} \right) \right) d^n x + \frac{\partial}{\partial x^\mu} \left( \xi^i \frac{\partial L}{\partial z^i_\mu} \right) d^n x,
\end{aligned}$$

where in the last equality we have separated the last summand in the second equation with the objetive of integrating by parts. Then, integrating $(j^1\phi)^* £_{\xi^{(1)}}\mathcal{L}$ and applying Stokes' Theorem, we obtain

$$\begin{aligned}
0 = \int_D (j^1\phi)^* £_{\xi^{(1)}}\mathcal{L} &= \int_D \xi^i \left( \frac{\partial L}{\partial y^i} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial z^i_\mu} \right) \right) d^n x + \int_{\partial D} \xi^i \frac{\partial L}{\partial z^i_\mu} d^{n-1} x_\mu \qquad (1.2) \\
&= \int_D \xi^i \left( \frac{\partial L}{\partial y^i} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial z^i_\mu} \right) \right) d^n x, \qquad (1.3)
\end{aligned}$$

where

$$d^{n-1} x_\mu := \iota_{\frac{\partial}{\partial x^\mu}} d^n x$$

and in the last equality we have used $\xi^i\big|_{\partial D} = 0$. Since Eq. (1.3) must hold for all possible values of $\xi^i$, we must have

$$\frac{\partial L}{\partial y^i} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial z^i_\mu} \right) = 0,$$

which are the Euler-Lagrange equations of the theory.

## 1.2 The Intrinsic Field equations

We can try to mimic the calculation performed in the previous section in an intrinsic way:

$$£_{\xi^{(1)}}\mathcal{L} = \iota_{\xi^{(1)}} d\mathcal{L} + d\iota_{\xi^{(1)}}\mathcal{L},$$

and integrating

$$\int_D (j^1\phi)^* \pounds_{\xi^{(1)}} \mathcal{L} = \int_D (j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L} + \int_D (j^1\phi)^* d\iota_{\xi^{(1)}} \mathcal{L} \tag{1.4}$$

$$= \int_D (j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L} + \int_{\partial D} (j^1\phi)^* \iota_{\xi^{(1)}} \mathcal{L} \tag{1.5}$$

$$= \int_D (j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L} = 0, \tag{1.6}$$

for every vertical vector field $\xi$. This, of course, does *not* imply

$$(j^1\phi)^* \iota_{\xi^{(1)}} d\mathcal{L} = 0,$$

for every vertical vector field $\xi \in \mathfrak{X}(Y)$ (which would be a possible way of expressing the field equations). A way of obtaining this intrinsic extremal condition would be to have Eq. (1.6) not only for lifts of vector fields, but for all vector fields. This, however, is simply noy true for $\mathcal{L}$. Nevertheless, we can try to modify the form $\mathcal{L}$ to certain form $\Theta_{\mathcal{L}}$ satisfying

$$(j^1\phi)^* \Theta_{\mathcal{L}} = (j^1\phi)^* \mathcal{L},$$

for all sections $\phi$ and such that $\phi$ is extremal if and only if (notice that Eq. (1.6) would hold also for $\Theta_{\mathcal{L}}$)

$$\int_D (j^1\phi)^* \iota_\xi d\Theta_{\mathcal{L}} = 0,$$

for all vector fields $\xi \in \mathfrak{X}(J^1\pi)$. As explained, this would imply

$$(j^1\phi)^* \iota_\xi d\Theta_{\mathcal{L}} = 0,$$

for all vector fields $\xi \in \mathfrak{X}(J^1\pi)$, which would give an intrinsic version of the field equations.

What are the conditions that $\Theta_{\mathcal{L}}$ needs to satisfy? We know that the extremal conditions holds for prolongations of vector fields $\xi^{(1)}$ which have the local expression

$$\xi^{(1)} = \xi^i \frac{\partial}{\partial y^i} + \left( \frac{\partial \xi^i}{\partial x^\mu} + \frac{\partial \xi^i}{\partial y^j} z^j_\mu \right) \frac{\partial}{\partial z^i_\mu}$$

Also, notice that $d\Theta_{\mathcal{L}}$ would be an $(n+1)$-form, and, therefore, for vector fields of the form

$$(j^1\phi)_* \xi_X$$

the condition holds trivially since $(j^1\phi)^* \iota_\xi d\Theta_{\mathcal{L}} = \iota_{\xi_X} (j^1\phi)^* \Theta_{\mathcal{L}} = 0$. The local expression of such a vector field would be

$$\xi = \phi_* \xi_X = \xi^\mu_X \left( \frac{\partial}{\partial x^\mu} + z^i_\mu \frac{\partial}{\partial y^i} + \frac{\partial^2 \phi^i}{\partial x^\mu \partial x^\nu} \frac{\partial}{\partial z^i_\nu} \right).$$

So, in order to have

$$\int_D (j^1\phi)^* \iota_\xi d\Theta_{\mathcal{L}} = 0,$$

for all vector fields $\xi \in \mathfrak{X}(J^1\pi)$, we only need to impose it for vector fields generated by

$$\frac{\partial}{\partial z^i_\mu},$$

or rather, for vertical vector fields of the (affine, see Appendix B) bundle $J^1\pi \to Y$. Of course, to satisfy this condition it is enough to ask for every vertical vector field $\xi$ to satisfy

$$(j^1\phi)^* d\iota_\xi \Theta_{\mathcal{L}} = 0,$$

for every section. Let us recap the conditions for $\Theta_{\mathcal{L}}$ :

(a) For every section $\phi : X \to Y$,

$$(j^1\phi)^*\Theta_{\mathcal{L}} = (j^1\phi)^*\mathcal{L}.$$

(b) For every section $\phi : X \to Y$, and every vertical vector field $\xi$ of $J^1\pi \to Y$,

$$(j^1\phi)^* d\iota_\xi \Theta_{\mathcal{L}} = 0.$$

Fix a volume form $\eta$ on $X$ (we will later see that this construction does not depend on this choice). One possible way of guaranteeing[2] (a) is to define

$$\Theta_{\mathcal{L}} := \mathcal{L} + \alpha \circ S_\eta,$$

for certain 1-form $\alpha$. Here, $S_\eta$ denotes the vertical endomorphism. For our purposes (for more details, we refer to Appendix B), it only suffices to know that for each wolume form $\eta$ on $X$ there is a canonical $n$-form with values in the vertical distribution that has the local expression

$$S_\eta = (dy^i - z^i_\nu dx^\nu) \wedge d^{n-1}x_\mu \otimes \frac{\partial}{\partial z^i_\mu},$$

where $\eta = d^n x$, $\mathcal{L} = L_\eta d^n x = L_\eta \eta$. If $\alpha$ has the local expression

$$\alpha = A^\mu_i dz^i_\mu + B_i dy^i + C_\mu dx^\mu,$$

we have

$$\Theta_{\mathcal{L}} = L_\eta d^n x + A^\mu_i (dy^i - z^i_\nu dx^\nu) \wedge d^{n-1}x_\mu.$$

Then, a quick calculation shows that condition (b) is equivalent to

$$\frac{\partial L_\eta}{\partial z^i_\mu} = A^i_\mu$$

[2] $S_\eta$ satisfies $(j^1\phi)^* S_\eta = 0$, for every section $\phi : X \to Y$, as one can easily check in coordinates.

and, therefore, if we define $\alpha := dL_\eta$, we obtain the desired form

$$\Theta_{\mathcal{L}} = \mathcal{L} + (dL_\eta) \circ S_\eta.$$

Notice that if we choose a different volume form, $\omega = f\eta$, for $f \neq 0$, we have that the new Lagrangian function is related to the previous one by

$$L_\omega = \frac{L_\eta}{f}.$$

This implies

$$(dL_\omega) \circ S_\omega = (dL_\eta) \circ S_\eta$$

and we conclude that, even though $\Theta_{\mathcal{L}}$ is not characterized by the conditions (a) and (b), $\Theta_{\mathcal{L}}$ is a form that satisfies these two, and it is intrinsically defined.

**Definition 1.2.1** (Poincaré-Cartan form). $\Theta_{\mathcal{L}}$ is called **the Poincaré-Cartan form** of the theory.

All of this discussion can be summarized by:

**Theorem 1.2.1** (Intrinsic Euler-Lagrange equations). *Let $Y \to X$ be a fibered manifold and $\mathcal{L} : J^1\pi \to \bigwedge^n X$ be a Lagrangian. Define the action*

$$S_D[\phi] := \int_D \mathcal{L} \circ j^1\phi,$$

*for every oriented compact $n$-dimensional submanifold $D \subseteq X$, where the values of the sections on $\partial D$ are fixed. Then, a section $\phi : X \to Y$ extremizes the previous functional for all possible $D \subseteq X$ if and only if*

$$(j^1\phi)^* \iota_\xi d\Theta_{\mathcal{L}} = 0,$$

*for every vector field $\xi \in \mathfrak{X}(J^1\pi)$, where*

$$\Theta_{\mathcal{L}} = \mathcal{L} + (dL_\eta) \circ S_\eta,$$

*is the Poincaré-Cartan form of the theory.*

**Remark 1.2.1.** Notice that in Theorem 1.2.1 we did not ask for $X$ to be orientable and, indeed, it is not necessary. Since we are interested in extremizing the action *locally*, we can choose an orientation for each $D$ beforehand (for the sake of integration). Also, since the choice of the volume form does not change the definition of $\Theta_{\mathcal{L}}$, we can get away with defining it locally.

In coordinates,

$$\Theta_{\mathcal{L}} = L d^n x + \frac{\partial L}{\partial z^i_\mu} (dy^i - z^i_\nu dz^\nu) \wedge d^{n-1} x_\mu \tag{1.7}$$

$$= \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1} x_\mu - \left( \frac{\partial L}{\partial z^i_\mu} z^i_\mu - L \right) d^n x. \tag{1.8}$$

This rather sugestive way of writing $\Theta_{\mathcal{L}}$ will be clarified in the following section.

**Definition 1.2.2.** Define **the multisymplectic form** of the theory as

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}}.$$

Locally,

$$\Omega_{\mathcal{L}} = -d\left(\frac{\partial L}{\partial z^i_\mu}\right) \wedge dy^i \wedge d^{n-1}x_\mu + d\left(\frac{\partial L}{\partial z^i_\mu} z^i_\mu - L\right) \wedge d^n x. \tag{1.9}$$

**Example 1.2.1.** *Following our example with Classical Mechanics, the Poincaré-Cartan form on* $J^1\pi = TY \times \mathbb{R}$ *has the local expression*

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial \dot{q}^i} dq^i - \left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right) dt,$$

*which justifies its name, since in this case it is the classical Poincaré-Cartan form (see [BSF88]). The multisymplectic form in this case is*

$$\Omega_{\mathcal{L}} = -d\left(\frac{\partial L}{\partial \dot{q}^i}\right) \wedge dq^i + d\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right) \wedge dt.$$

For future considerations, it will be useful to generalize the equation of Theorem 1.2.1 to sections

$$\sigma : X \to J^1\pi$$

satisfying

$$\sigma^* \iota_\xi \Omega_{\mathcal{L}} = 0, \tag{1.10}$$

for every vector field $\xi \in \mathfrak{X}(J^1\pi)$. Eq. (1.10) will be refered as the **De Donder equation**.

A natural question to ask is whether every solution of De Donder equations is the lift of a solution of the Euler-Lagrange equations, that is, wether a section $\sigma : X \to J^1\pi$ satisfying

$$\sigma^* \iota_\xi \Omega_{\mathcal{L}} = 0,$$

for every vector field $\xi$ on $J^1\pi$ necessarily is of the form

$$\sigma = j^1\phi,$$

for certain section $\phi : X \to Y$. This is the case for a particular family of Lagrangian densities:

**Definition 1.2.3** (Regular Lagrangian)**.** Let $Y \to X$ be a fibered manifold and

$$\mathcal{L} : J^1\pi \to \bigwedge^n X$$

a Lagrangian density. $\mathcal{L}$ will be called **regular** if for some (any) set of coordinates $(x^\mu, y^i, z^i_\mu)$, the matrix

$$\left(\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu}\right)_{(i,\mu),(j,\nu)}$$

is non-singular. Here,

$$\mathcal{L} = L d^n x,$$

for some set of coordinates $(x^\mu)$.

**Proposition 1.2.1.** *Let $\mathcal{L}$ be a regular Lagrangian. Then, for every solution of De Donder equation $\sigma : X \to J^1\pi$, there exists an unique section $\phi : X \to Y$ such that*

$$\sigma = j^1\phi.$$

*Proof.* Uniqueness is clear. To prove existence, we will do it in coordinates. Let $\sigma$ be a solution of the De Donder equation with local expression

$$\sigma^* x^\mu = x^\mu;$$
$$\sigma^* y^i = \sigma^i;$$
$$\sigma^* x^i_\mu = \sigma^i_\mu.$$

We will conclude the proof once we show that we have

$$\sigma^i_\mu = \frac{\partial \sigma^i}{\partial x^\mu}.$$

Let $\xi = \frac{\partial}{\partial z^j_\nu}$. Then,

$$\iota_\xi \Omega_{\mathcal{L}} = -\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} dy^i \wedge d^{n-1}x_\mu + z^i_\mu \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} d^n x.$$

Since $\sigma^* \iota_\xi \Omega_{\mathcal{L}} = 0$, we must have

$$0 = \sigma^* \iota_\xi \Omega_{\mathcal{L}} = -\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \frac{\partial \sigma^i}{\partial x^\mu} d^n x + \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \sigma^i_\mu d^n x,$$

which gives

$$\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \left( \sigma^i_\mu - \frac{\partial \sigma^i}{\partial x^\mu} \right) = 0,$$

for every $(j, \nu)$. Since $\mathcal{L}$ is regular, this implies

$$\sigma^i_\mu = \frac{\partial \sigma^i}{\partial x^\mu},$$

which concludes the proof. □

Proposition 1.2.1 implies that, for regular Lagrangians, we can look for solutions defining a connection on the fibered manifold

$$J^1\pi \to X,$$

or defining a decomposable $n$-multivector field transversal to the vertical distribution. Then, the field $\phi : X \to Y$ that extremizes the action would just be the field satisfying

$$\sigma = j^1\phi,$$

for one integral section $\sigma : X \to J^1\pi$ of one of the mentioned objects. We will deal with these ideas in the next section.

## 1.3 The Hamiltonian Formalism

The Hamiltonian formalism is obtained via the **Legendre transformation**, which is defined as follows. Notice that the Poincaré-Cartan form

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x_\mu - \left( \frac{\partial L}{\partial z^i_\mu} z^i_\mu - L \right) d^n x$$

is semi-basic[3] in the fibered manifold

$$\pi_{1,0} : J^1\pi \to Y.$$

Therefore, to each point $j^1\phi \in J^1\pi$, there corresponds an unique form

$$\mathrm{Leg}_{\mathcal{L}}(j^1\phi) \in \bigwedge^n T^*_{\phi(y)} Y$$

such that

$$\Theta_{\mathcal{L}}(j^1\phi) = \pi^*_{1,0} \,\mathrm{Leg}_{\mathcal{L}}(j^1\phi).$$

Of course, in coordinates,

$$\mathrm{Leg}_{\mathcal{L}}(x^\mu, y^i, z^i_\mu) = \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x_\mu - \left( \frac{\partial L}{\partial z^i_\mu} z^i_\mu - L \right) d^n x.$$

This is the Legendre transformation

$$J^1\pi \xrightarrow{\mathrm{Leg}_{\mathcal{L}}} \bigwedge^n Y.$$

Observe that $\mathrm{Leg}_{\mathcal{L}}(j^1\phi)$ vanishes when contracted with two vertical vector fields of $Y \xrightarrow{\pi} X$ so, if we define

$$\bigwedge^n_2 Y = \left\{ \alpha_y \in \bigwedge^n Y \;:\; \iota_{\xi_1 \wedge \xi_2} \alpha = 0, \text{ if } \pi_* \xi_i = 0 \right\},$$

we have

$$J^1\pi \xrightarrow{\mathrm{Leg}_{\mathcal{L}}} \bigwedge^n_2 Y.$$

Given coordinates $(x^\mu, y^i, p, p^\mu_i)$ in $\bigwedge^n_2 Y$ representing the form

$$p d^n x + p^\mu_i dy^i \wedge d^{n-1} x_\mu,$$

this map takes the local expression

$$\begin{aligned} \mathrm{Leg}^*_{\mathcal{L}}\, x^\mu &= x^\mu; \\ \mathrm{Leg}^*_{\mathcal{L}}\, y^i &= y^i; \\ \mathrm{Leg}^*_{\mathcal{L}}\, p &= L - \frac{\partial L}{\partial z^i_\mu} z^i_\mu; \\ \mathrm{Leg}^*_{\mathcal{L}}\, p^\mu_i &= \frac{\partial L}{\partial z^i_\mu}. \end{aligned}$$

[3] That is, $\iota_\xi \Theta_{\mathcal{L}} = 0$, for every vertical vector field $\xi$.

There is a canonical form defined on $\bigwedge_2^n Y$ (in fact, on $\bigwedge^n Y$).

**Definition 1.3.1.** For $\alpha \in \bigwedge_2^k Y$ define the **Louville form** as

$$\Theta_Y\Big|_\alpha(v_1, \dots, v_n) := \alpha(\tau_* v_1, \dots, \tau_* v_n),$$

where $\tau : \bigwedge_2^n Y \to Y$ is the natural proyection. Finally, define the **canonical multisymplectic form** as

$$\Omega_Y := -d\Theta_Y.$$

In coordinates,

$$\Theta = p d^n x + p_i^\mu dy^i \wedge d^{n-1}x_\mu,$$
$$\Omega_Y = -dp \wedge d^n x - dp_i^\mu \wedge d^{n-1}x_\mu.$$

And, clearly, we have

**Proposition 1.3.1.** *The Legendre transformation satisfies*

$$\mathrm{Leg}_{\mathcal{L}}^* \Theta_Y = \Theta_{\mathcal{L}}, \; \mathrm{Leg}_{\mathcal{L}}^* \Omega_Y = \Omega_{\mathcal{L}}.$$

The presentation we have chosen in the text in not the usual. The standard way of defining the Poincaré Cartan form $\Theta_{\mathcal{L}}$ and the multisymplectic form $\Omega_{\mathcal{L}}$ is through Proposition 1.3.1 with the following definition of $\mathrm{Leg}_{\mathcal{L}}$ which, in our case, needs to be proved.

The jet bundle $J^1\pi$ of the fibered manifold $\pi : Y \to X$ defines an affine bundle over $Y$ modeled by the vector bundle (esentially, modelled by linear maps $T_{\pi(y)}X \to \ker d_y\pi \subseteq T_Y Y$)

$$\pi^*(T^*X) \otimes \ker d\pi \to Y.$$

Then, one defines the Legendre transformation as a map from $J^1\pi$ to its "dual"

$$J^1\pi \xrightarrow{\widetilde{\mathrm{Leg}}_{\mathcal{L}}} \mathrm{Aff}(J^1\pi, \pi^*\left(\bigwedge^n X\right)) \; .$$

Although $\mathrm{Aff}(J^1\pi, \pi^*\left(\bigwedge^n X\right))$ and $\bigwedge_2^n Y$ are not technically the same manifold, they are canonically diffeomorphic by the map that assigns to each form $\alpha \in \bigwedge_2^n Y$, the affine map

$$j^1\phi \mapsto (j^1\phi)^*\alpha.$$

Then, we have the following alternative definition

**Proposition 1.3.2.** *Under the above identifaction (that is, interpreting each form* $\mathrm{Leg}_{\mathcal{L}}(j^1\phi)$ *as an affine map), we have that, for* $j^1\phi, j^1\psi$ *in the same fiber of* $J^1\pi \to Y$,

$$\langle \mathrm{Leg}_{\mathcal{L}}(j^1\phi), j^1\psi\rangle = \mathcal{L} + \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(\mathcal{L}(j^1\phi + t(j^1\psi - j^1\phi))\right)^{4}.$$

[4] We are making abuse of notation here, notice that $J_y^1\pi$ (some fiber of $J^1\pi \to Y$) is an affine space and, therefore, $j^1\psi - j^1\phi$ can be thought as a vector, which we can multiply by any scalar $t$, and then add to $j^1\phi$ to obtain another point in $J^1\pi$.

*Proof.* We will prove it locally. Indeed, defining on $\mathrm{Aff}(J^1\pi, \pi^*\left(\bigwedge^n X\right))$ coordintes $(x^\mu, y^i, p, p_i^\mu)$ representing the affine map

$$w^i_\mu \mapsto (p + p_i^\mu w^i_\mu) d^n x,$$

the local expression of the identification with $\bigwedge_2^n Y$ is the identity. Now, we have

$$L d^n x + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L(z^i_\mu + t(w^i_\mu - z^i_\mu)) d^n x = L d^n x + \frac{\partial L}{\partial z^i_\mu}(w^i_\mu - z^i_\mu) d^n x,$$

which in coordinates is the map represented by $(x^\mu, y^i, p = L - \frac{\partial L}{\partial z^i_\mu} z^i_\mu, p_i^\mu = \frac{\partial L}{\partial z^i_\mu})$, which is exactly the Legendre transformation. □

The best possible scenario for a Legendre transformation would be to have a perfect bridge (or rather, a diffeomorphism) between the Lagrangian and Hamiltonian formulation. However, with our definition of $\mathrm{Leg}_\mathcal{L}$ this is not possible, since

$$\dim \bigwedge_2^n Y = \dim J^1\pi + 1 > \dim J^1\pi.$$

To salvage this, we can "ignore" the $p$ coordinate to obtain a transformation

$$(x^\mu, y^i, z^i_\mu) \mapsto (x^\mu, y^i, \frac{\partial L}{\partial z^i_\mu}),$$

which could deffine a diffeomorpishm. Intrinsically, this corresponds to quotienting

$$\bigwedge_2^n Y / \bigwedge_1^n Y,$$

where $\bigwedge_1^n Y$ is the space of all semi-basic forms on the fibered manifold $\pi : Y \to X$ (taking into account the identification with the affine maps, semi-basic forms are the constant affine maps).

**Definition 1.3.2** (Reduced Legendre transformation)**.** The reduced Legendre transformation is the unique map that closes the following diagram

$$\begin{array}{ccc} J^1\pi & \xrightarrow{\mathrm{Leg}_\mathcal{L}} & \bigwedge_2^n Y \\ & \searrow^{\mathrm{leg}_\mathcal{L}} & \downarrow \\ & & \bigwedge_2^n Y / \bigwedge_1^n Y \end{array}.$$

If we define

$$(J^1)^*\pi := \bigwedge_2^n Y / \bigwedge_1^n Y,$$

then, the reduced Legendre transformation is a map

$$J^1\pi \xrightarrow{leg_{\mathcal{L}}} (J^1)^*\pi,$$

which in coordinates reads

$$\begin{aligned}(leg_{\mathcal{L}})^* x^\mu &= x^\mu;\\ (leg_{\mathcal{L}})^* y^i &= y^i;\\ (leg_{\mathcal{L}})^* p_i^\mu &= \frac{\partial L}{\partial z_\mu^i}.\end{aligned}$$

This coordinate expression implies the following:

**Proposition 1.3.3.** *The reduced Legendre transformation $leg_{\mathcal{L}}$ defines a local diffeomorphism if and only if $\mathcal{L}$ is a regular Lagrangian.*

**Definition 1.3.3.** When the reduced Legendre transformation defines a *global* diffeomorphism, $\mathcal{L}$ is called a **hyper-regular Lagrangian**.

There are several reasons to use $\bigwedge_2^n Y$ over $(J^1)^*\pi$, and viceversa. The main advantage of $(J^1)^*\pi$ is the one already mentioned, we have the possibility of a perfect correspondence between the Lagrangian and Hamiltonian formalism, which will be made explicit in the theorem below. However, there is an important drawback, $(J^1)^*\pi$ does not carry a canonical multisymplectic structure, unlike $\bigwedge_2^n Y$. So, to define the forms that intrinsically characterize the fields we are looking form, we need a section

$$\hbar : (J^1)^*\pi \to \bigwedge_2^n Y^{5},$$

in order to define

$$\begin{aligned}\Theta_\hbar &:= \hbar^*\Theta_Y;\\ \Omega_\hbar &:= \hbar^*\Omega_Y.\end{aligned}$$

The local expression of the Hamiltonian section $\hbar = \mathrm{Leg}_{\mathcal{L}} \circ (leg_{\mathcal{L}})^{-1}$ in canonical coordinates $(x^\mu, y^i, p, p_i^\mu)$ is

$$\begin{aligned}\hbar^* x^\mu &= x^\mu;\\ \hbar^* y^i &= y^i;\\ \hbar^* p &= L - \frac{\partial L}{\partial z_\mu^i} z_\mu^i =: -\mathcal{H};\\ \hbar^* p_i^\mu &= p_i^\mu.\end{aligned}$$

[5] which generally is going to be $\hbar = \mathrm{Leg}_{\mathcal{L}} \circ (leg_{\mathcal{L}})^{-1}$, for some hyper-regular Lagrangian $\mathcal{L}$

So, locally,

$$\begin{aligned}\Theta_{\hbar} &= -\mathcal{H} d^n x + p_i^{\mu} dy^i \wedge d^{n-1} x_{\mu},\\ \Omega_{\hbar} &= d\mathcal{H} \wedge d^n x - dp_i^{\mu} \wedge dy^i \wedge d^{n-1} x_{\mu},\end{aligned}$$

which justifies the writing of $\Omega_{\mathcal{L}}$ in Eq. (1.8).

Going back to correspondence we have anticipated, the theory developed so far can be neatly summarized in the following theorem

**Theorem 1.3.1** (Correspondence Theorem between Lagrangian and Hamiltonian formalism)**.** *Let $\pi : Y \to X$ be a fibered manifold and $\mathcal{L} : J^1\pi \to \bigwedge^n X$ be a hyper-regular Lagrangian. Define*

$$S_D[\phi] := \int_D \mathcal{L} \circ j^1\phi,$$

*for every compact oriented $n$-dimensional submanifold of $X$ and every section $\phi : X \to Y$. Then, for a section $\phi : X \to Y$, the following are equivalent*

*(a) $\phi$ extremizes $S_D$, for every compact $n$-dimensional submanifold $D \subseteq X$;*

*(b) For every vector field $\xi \in \mathfrak{X}(J^1\pi)$,*

$$(j^1\phi)^* \iota_{\xi} \Omega_{\mathcal{L}} = 0;$$

*(c) $\phi = \pi_{1,0} \circ \sigma$, for an unique section $\phi : X \to J^1\pi$ satisfying*

$$\sigma^* \iota_{\xi} \Omega_{\mathcal{L}} = 0,$$

*for every vector field $\xi \in \mathfrak{X}(J^1\pi)$, where $\pi_{1,0} : J^1\pi \to Y$ is the natural projection;*

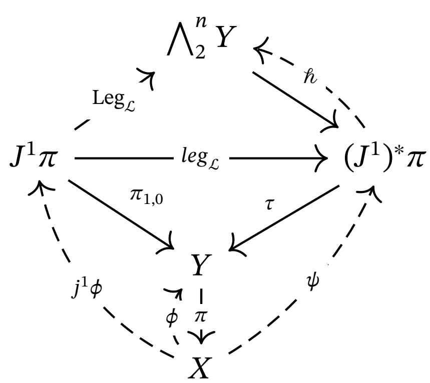
*(d) $\phi = \tau \circ \psi$, for an unique section*

$$\psi : X \to (J^1)^*\pi$$

*satisfying*

$$\psi^* \iota_{\xi} \Omega_{\hbar} = 0,$$

*for every vector field $\xi \in \mathfrak{X}((J^1)^*\pi)$, where $\hbar := \mathrm{Leg}_{\mathcal{L}} \circ (leg_{\mathcal{L}})^{-1}$ and $\tau : (J^1)^*\pi \to Y$ is the natural projection.*



Locally, the differential equations that a section

$$\psi : X \to (J^1)^*\pi$$

needs to satisfy in order to extremize the action functional are the Hamilton-De Donder-Weyl equations

$$\frac{\partial \psi^i}{\partial x^\mu} = \frac{\partial \mathcal{H}}{\partial p_i^\mu};$$
$$\sum_\mu \frac{\partial \psi_i^\mu}{\partial x^\mu} = -\frac{\partial \mathcal{H}}{\partial y^i},$$

where

$$\mathcal{H} = \frac{\partial L}{\partial z_\mu^i} z_\mu^i - L$$

is the **Hamiltonian**.

Now, as we have already mentioned, it might be interesting to work with $\bigwedge_2^n Y$ instead of working with $(J^1)^*\pi$. Then, the following question arises:

Can we prove a result similar to Theorem 1.3.1 in $\bigwedge\limits_2^n Y$?

Fortunately, the answer is yes, and to prove it we are going to identify solutions as integral sections of descomposable multivector fields (which will be called *Hamiltonian* multivector fields).

Let us first study how to give a different notion of solution on $(J^1)^*\pi$ (or on $J^1\pi$). Given an $n$-dimensional distribution $\Delta$ transverse to the fibers of

$$(J^1)^*\pi \to X,$$

a section

$$\psi : X \to (J^1)^*\pi$$

is called an **integral section** of $\Delta$ if

$$\psi_* T_x X = \Delta_{\psi(x)},$$

for all $x \in X$. One possible way of defining such a distribution would be via a locally decomposable multivector field of order $n$,

$$U = X_1 \wedge \cdots \wedge X_n,$$

where $X_i$ is a vector field locally defined, representing the distribution

$$\Delta = \langle X_1, \dots, X_n \rangle.$$

Then, by Theorem 1.3.1, it is clear that (for integrable $\Delta$), every section defines a critical point of the variational problem if and only if

$$\iota_U \Omega_{\hbar} = 0.$$

Notice that $U$ and $fU$ represent the same distribution and the extremal condition does not depend on $f$, for any nowhere vanishing $f \in C^\infty((J^1)^*\pi)$.

Now, a section which, for the sake of simplicity, we can think of it arising from an hyper-regular Lagrangian,

$$\hbar : (J^1)^*\pi \to \bigwedge_2^n Y$$

can be identified as a level set of some (which can be chosen in a canonical way, see [RW19]) function

$$H : \bigwedge_2^n Y \to \mathbb{R}.$$

Then, a multivector field $U$ on $(J^1)^*\pi$ defines a solution for the variational problem if and only if

$$\iota_U \Omega_{\hbar} = 0,$$

or rather, if and only if

$$\iota_{\hbar_* U} \Omega_Y = 0$$

on the corresponding level set of $H$. This implies (whenever $\hbar_* U$ is defined)

$$\iota_{h_* U} \Omega_Y = \alpha \, dH,$$

for certain function $\alpha$. Since $U$ is defined up to multiplications by constants, we can modify (ignoring some technical issues, because we could have $\alpha = 0$) so that

$$\iota_U \Omega_Y = dH.$$

Extending this notion to $\bigwedge_2^n Y$, we get the following definition, which will be one of the main objets we will study in the following chapters.

**Definition 1.3.4** (Hamiltonian $n$-multivector field)**.** A $n$-multivector field $U$ on $\bigwedge_2^n Y$ is called **Hamiltonian** if

$$\iota_U \Omega_Y = dH,$$

for certain funtion $H$ on $\bigwedge_2^n Y$.

## 1.4 Time-independent Classical Mechanics and symplectic geometry

As we have seen, Classical Mechanics can be interpreted as a Classical Field Theory with the bundle

$$Q \times \mathbb{R} \xrightarrow{\pi} \mathbb{R},$$

where $Q$ is the configuration manifold. With the canonical volume form $\eta = dt$ on $\mathbb{R}$, a Lagrangian density

$$\mathcal{L} : J^1\pi = TQ \times \mathbb{R} \to T^*\mathbb{R}$$

is characterized by the **Lagrangian function** $L : TQ \times \mathbb{R} \to \mathbb{R}$. As we proved, critical sections (or curves)

$$\gamma : \mathbb{R} \to Q \times \mathbb{R}$$

are characterized by

$$\iota_{\dot\gamma}\Omega_{\mathcal{L}} = 0,$$

where

$$\Omega_{\mathcal{L}} = -d\left(\frac{\partial L}{\partial \dot q^i}\right) \wedge dq^i + d\left(\frac{\partial L}{\partial \dot q^i}\dot q^i - L\right) \wedge dt.$$

In the case that $L$ is time-independent, that is, when it can be identified as a function

$$L : TQ \to \mathbb{R},$$

it is natural to try to rewrite the extremal condition of a section

$$\gamma(t) = (\alpha(t), t)$$

in terms of $\alpha$. Since

$$j^1\gamma = (\alpha, \dot\alpha, t),$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t} j^1\gamma = \ddot\alpha + \frac{\partial}{\partial t}.$$

Therefore, $\alpha$ defines an extremal if and only if

$$0 = \iota_{j^1\gamma}\Omega_{\mathcal{L}} = \iota_{\ddot\alpha}\left(-d\left(\frac{\partial L}{\partial \dot q^i}\right) \wedge dq^i\right) + d\left(\frac{\partial L}{\partial \dot q^i}\dot q^i - L\right) \cdot \ddot\alpha dt - d\left(\frac{\partial L}{\partial \dot q^i}\dot q^i - L\right).$$

This equation can be splitted in two

$$\iota_{\ddot\alpha}\left(-d\left(\frac{\partial L}{\partial \dot q^i}\right) \wedge dq^i\right) = d\left(\frac{\partial L}{\partial \dot q^i}\dot q^i - L\right); \tag{1.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q^i}\dot q^i - L\right) = 0 \tag{1.12}$$

so, defining the **symplectic form**

$$\omega_L := -d\left(\frac{\partial L}{\partial \dot q^i}\right) \wedge dq^i,$$

a curve $\alpha : \mathbb{R} \to Q$ is an extremizer of the action if and only if

$$\iota_{\ddot\alpha}\omega_L = dE, \tag{1.13}$$

where $E$ is the classical Energy

$$E = \frac{\partial L}{\partial \dot q^i}\dot q^i - L.$$

Notice that Eq. (1.12) implies that $E$ is constant along the solutions. When $L$ is regular (recall that this means that the matrix $\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)_{ij}$ is non-degenerate), then $(TQ, \omega_L)$ defines a symplectic manifold, that is,

$$\iota_v \omega_L = 0$$

if and only if $v = 0$. Similar to how the multisymplectic structure on $J^1\pi$ can be obtained via the Legendre transformation, we can obtain the symplectic structure on $TL$ as well through the map

$$\mathbb{F}L : TQ \to T^*Q, \; \dot{q}^i \mapsto \frac{\partial L}{\partial \dot{q}^i}$$

and the canonical symplectic form on $T^*Q$. The discussion so far implies that we can identify a time-independent Clasiscal Mechanical System with a symplectic manifold:

**Definition 1.4.1** (symplectic manifold)**.** A **symplectic manifold** is a pair $(M, \omega)$, where $M$ is a $2n$-dimensional manifold and $\omega$ is a closed, non-degenerate, 2-form.

**Example 1.4.1.** *Fixed a configuration manifold Q, its cotangent bundle $T^*Q$ admits a canonical symplectic structure defined as follows. First, define the* ***Liouville*** 1*-form $\lambda$ by*

$$\lambda_Q|_\alpha(v) := \alpha(\pi_* v),$$

*for $\alpha \in T^*Q$, $v \in T_\alpha T^*Q$. Then,*

$$\omega_Q := -d\lambda_Q$$

*defines a symplectic structure on Q. The sympelctic form $\omega_Q$ is the unique form satisfying*

$$\alpha^*\omega_Q = -d\alpha,$$

*for every* 1*-form $\alpha : Q \to T^*Q$.*

We will now shift the focus to state the main theorems that we are going to generalize in the remaining of the text for multisymplectic manifolds. First, we need some notions of symplectic geometry.

Eq. (1.13) induces the following definition

**Definition 1.4.2** (Hamiltonian, locally Hamiltonian vector field)**.** Let $(M, \omega)$ be a symplectic manifold and $H \in C^\infty(M)$ be a function, which will be called a **Hamiltonian**. The **Hamiltonian vector field** associated to $H$ is the unique vector field $X_H$ satisfying

$$\iota_{X_H}\omega = dH.$$

A vector field $X \in \mathfrak{X}(M)$ will be called **locally Hamiltonian** if $\iota_{X_H}\omega$ is closed (opposed to exact, in the previous case).

Then, dynamics correspond to Hamiltonian vector fields for some $H$, the energy. The first result allows us to give a different interpretation of dynamics. More precisely, we will give an interpretation of (locally) Hamiltonian vector fields as Lagrangian submanifolds:

**Definition 1.4.3** (Isotropic, Lagrangian submanifold)**.** Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. Then a submanifold $i : N \hookrightarrow M$ is called **isotropic** if $i^*\omega = 0$. If a submanifold is isotropic, necessarily $\dim N \leq n$. Then, a **Lagrangian** submanifold is an isotropic submanifold $N$ of maximal dimension $n$.

The first observation we need to make is that we can endow $TM$ with a symplectic structure. Indeed, the symplectic form $\omega$ yields a diffeomorphism induced by contraction

$$TM \xrightarrow{\flat} T^*M; \ v \mapsto \iota_v\omega.$$

Then, we can define a canonical symplectic form on $TM$ by

$$\widetilde{\omega}_M := \flat^*\omega_M,$$

where $\omega_M$ is the canonical symplectic form on $T^*M$. We have the following result:

**Proposition 1.4.1.** *A vector field $X : M \to TM$ is locally Hamiltonian if and only if it defines a Lagrangian submanifold in $(TM, \widetilde{\omega}_M)$.*

*Proof.* Indeed,

$$X^*\widetilde{\omega}_M = -d\iota_X\omega,$$

which yields the result. □

The last one of the results has to do with coisotropic submanifolds. The main reason to consider this kind of submanifolds was observed by Dirac in [Dir58], where he presented his theory of constraints. We now present a (very) brief summary of the main ideas. Given a Lagrangian

$$L : TQ \to \mathbb{R}$$

characterizing some Classical Mechanical system, if we aim to quantize it[6], the natural way is going to the Hamiltonian formalism via the Legendre transformation $\mathbb{F}L : TQ \to T^*Q$ and performing canonical quantization. If our original Lagrangian is regular, this procedure works fine (at least, fine until quantizing) but, in any other scenario, we get a submanifold of constraints

$$\mathbb{F}L(TQ) \subseteq T^*Q.$$

Then, a quantization procedure would have to incorporate this set of constraints as a subspace of the Hilbert space corresponding to $Q$. Now, quantizing does not only assigns a Hilbert space, but (ideally) a Hermitian operator to each observable $f \in C^\infty(T^*Q)$. This map, which we will denote by $f \mapsto \hat{f}$, should satisfy the fundamental equality

$$[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}},$$

where $\{f, g\}$ denotes the Poisson bracket, defined as

$$\{f, g\} := \omega_Q(X_f, X_g).$$

[6]that is, assigining to it a Hilbert space and a Hamiltonian Hermitian operator $\widehat{H}$

What Dirac proposed was that in order to quantize the constraints, if a function $f$ was constant in $\mathbb{F}L(TQ)$, then the corresponding operator $\hat{f}$ should be a multiple of the identity when restricted to the subspace corresponding to $\mathbb{F}L(TQ)$. This, of course, would imply

$$[\hat{f}, \hat{g}] = 0,$$

for any functions $f, g \in C^\infty(T^*Q)$ which are constant on $\mathbb{F}L(TQ)$. So, Dirac proposed that the correct constraints to quantize are the ones satisfying

$$\{f, g\} = 0,$$

for any functions constant on the submanifold.

**Definition 1.4.4** (Coisotropic submanifold)**.** Let $(M, \omega)$ be a symplectic manifold. Then, a coisotropic submanifold $i : N \hookrightarrow M$ is called **coisotropic** if for any functions $f, g \in C^\infty(M)$ which are constant on $N$, $\{f, g\} = 0$ on $N$.

This is not the usual definition. However, its is equivalent to the one usually given which is in terms of symplectic algebra.

**Proposition 1.4.2.** *$N$ is coisotropic if and only if it satisfies*

$$(T_qN)^\perp \subseteq T_qN,$$

*for each $q \in N$, where*

$$(T_qN)^\perp = \{v \in T_qN;\ \omega_q(v, w) = 0, \forall w \in T_qN\}.$$

As an immediate observation, a coisotropic submanifold necessarily satisfies $\dim N \geq n$.

Now that we have understood the meaning of coisotropic submanifolds, let us end this chapter stating the main result regarding them, and the one that we will generalize (the proof will be given later on).

**Theorem 1.4.1** (Coisotropic reduction in symplectic geometry)**.** *Let $(M, \omega)$ be a symplectic manifold, $i : N \hookrightarrow M$ be a coisotropic submanifold, and $j : L \hookrightarrow M$ be a Lagrangian submanifold that has clean intersection with $N$. Then, $TN^\perp$ is an integrable distribution and determines a foliation $\mathcal{F}$ of maximal integral leaves. Suppose that the quotient space $N/\mathcal{F}$ admits an smooth manifold structure such that the canonical projection*

$$\pi : N \to N/\mathcal{F}$$

*defines a submersion. Then there exists an unique symplectic form on $N/\mathcal{F}$, $\omega_N$ compatible with $\omega$ in the following sense*

$$\pi^*\omega_N = i^*\omega.$$

*Furthermore, if $\pi(N \cap L)$ is a submanifold, it is Lagrangian in $(N/\mathcal{F}, \omega_N)$.*

For the usual definitions and the proofs of their equivalences to the ones given in this section, see Appendix A.

# Chapter 2

# Multisymplectic geometry

In Chapter 1, we saw that the solutions to a variational problem can be studied geometrically with several structures $(J^1\pi, \Omega_{\mathcal{L}})$, $((J^1)^*\pi, \Omega_{\hbar})$, or $(\bigwedge_2^n Y, \Omega_Y)$. In this chapter we generalize this idea to a manifold together with a closed form $(M, \omega)$ (even though the forms defined on te previous spaces are exact, most results only use closedness since they are of local nature).

We begin the study of these kind of structures in Section 2.1 by linearizing the problem to a vector space together with a form $(V, \omega)$, where we give the main examples and characterize them. Later, in Section 2.2, we study the main examples of multisymplectic manifolds and we state a Darboux Theorem for them[1].

## 2.1 Multisymplectic vector spaces

**Definition 2.1.1** (Multisymplectic vector space)**.** A **multisymplectic vector space** of order $k$ is a pair $(V, \omega)$, where $\omega \in \bigwedge^{k+1} V^*$. The multisymplectic vector space and the form will be called **non-singular** or **regular** if the map given by contraction

$$V \xrightarrow{\flat_1} \bigwedge^k V; \;\; v \mapsto \iota_v\omega$$

defines a monomorphism, that is, $\iota_v\omega = 0$ if and only if $v = 0$.

**Observation 2.1.1.** This terminology is not standard. In the literature, an arbitrary form $\omega \in \bigwedge^k V^*$ is usually called *pre-multisymplectic*, but we choose this terminology for the sake of simplicity. We prefer this general aproach because in Chapter 3, "singular" multisymplectic manifolds (what we simply call multisymplectic) appear naturally. Nevertheless, all the definitions given in the text coincide with the usual definitions when $\omega$ is non-degenerate.

The isomorphism of multisymplectic vector spaces is given by the following definition.

**Definition 2.1.2** (Multisymplectomorphism)**.** Let $(V_1, \omega_1)$, $(V_2, \omega_2)$ be multisymplectic vector spaces. A **multisymplectomorphism** between $(V_1, \omega_1)$ and $(V_2, \omega_2)$ is a linear isomorphism

$$f : V_1 \to V_2$$

[1] This meaning a local characterization of this type of manifolds

satisfying

$$f^*\omega_2 = \omega_1.$$

**Example 2.1.1.** *Let $L$ be a vector space and take $V := L \oplus \bigwedge^k L$ with $k \leq \dim V$. Define the $(k+1)$-form*

$$\Omega_L((v_1, \alpha_1), \dots, (v_{k+1}, \alpha_{k+1})) := \sum_{j=1}^{k+1} \alpha_j(v_1, \dots, \hat{v}_j, \dots, v_{k+1}),$$

*where $\hat{v}_j$ means that the $j$th-vector is missing. Then, $\Omega_L$ is a regular multisymplectic form and, thus, $(V, \Omega_L)$ is a regular multisymplectic vector space.*

Similar to the notion of orthogonal in symplectic vector spaces, we can define a (now indexed) version in multisymplectic vector spaces.

**Definition 2.1.3** (Multisymplectic orthogonal)**.** Let $(V, \omega)$ be a multisymplectic vector space of order $k$, $W \subseteq V$ be a subspace and $1 \leq j \leq k$. Define the **$j$th-orthogonal** to $W$ as the subspace

$$W^{\perp,j} = \{v \in V \ : \ \iota_{v \wedge w_1 \wedge \cdots \wedge w_j}\omega = 0, \forall w_1, \dots, w_j \in W\}.$$

It can be easily proved that the $j$th-orthogonal satisfies the following properties:

**Proposition 2.1.1.** *Let $(V, \omega)$ be a multisymplectic vector space of order $k$. Then,*

*a)* $\{0\}^{\perp,j} = V$ *for all* $1 \leq j \leq k$;

*b)* $V^{\perp,1} = \ker \flat_1$;

*c)* $(W_1 + W_2)^{\perp,j} \subseteq W_1^{\perp,j} \cap W_2^{\perp,j}$, *for all* $1 \leq j \leq k$, $W_1, W_2 \subseteq V$ *subspaces;*

*d)* $W_1^{\perp,j} + W_2^{\perp,j} \subseteq (W_1 \cap W_2)^{\perp,j}$ *for all* $1 \leq j \leq k$, $W_1, W_2 \subseteq V$ *subspaces;*

*e)* $(W_1 + W_2)^{\perp,1} \subseteq W_1^{\perp,1} \cap W_2^{\perp,1}$, *for all* $W_1, W_2 \subseteq V$ *subspaces.*

In order to generalize the definitions of isotropic, coisotropic and Lagrangian (which have an indexed analoge), we notice that $\ker \flat_1 \subseteq W^{\perp,j}$, for every possible index $j$. Therefore, in the generalized definition, we must ask for inclusions up to $\ker \flat_1$.

**Definition 2.1.4** ($j$-isotropic, $j$-coisotropic, $j$-Lagrangian, non-degenerate)**.** Let $(V, \omega)$ be a multisymplectic vector space of order $k$. A subspace $W \subseteq V$ will be called

a) **$j$-isotropic**, if $W \subseteq W^{\perp,j}$;

b) **$j$-coisotropic**, if $W^{\perp,j} \subseteq W + \ker \flat_1$;

c) **$j$-Lagrangian**, if $W = W^{\perp,j} + \ker \flat_1$;

d) **non-degenerate**, if $W \cap W^{\perp,1} = 0$.

**Observation 2.1.2.** Notice that when $\omega$ is regular, $\ker \flat_1 = 0$, and we recover the standard definitions of $j$-isotropic, $j$-coisotropic, and $j$-Lagrangian.

**Proposition 2.1.2.** *Let $(V, \omega)$ be a multisymplectic vector space of order $k$. Then, a subspace $i : W \to V$ ($i$ being the natural inclusion) is $k$-isotropic if and only if*

$$i^*\omega = 0.$$

*Proof.* $W$ is $k$-isotropic if and only if

$$\omega(w_1, \dots, w_{k+1}) = 0,$$

for every $w_1, \dots, w_{k+1} \in W$ or, equivalently, $i^*\omega = 0$. □

**Example 2.1.2.** *Let $L$ be a vector space and $\mathcal{E} \subseteq L$ be a proper subspace. For $r \geq 0$, $k \leq \dim L$, define*

$$\bigwedge_r^k L^* := \left\{ \alpha \in \bigwedge^k L^* : \iota_{v_1 \wedge \cdots \wedge v_r} \alpha = 0, \forall v_1, \dots, v_r \in \mathcal{E} \right\}.$$

*Notice that, if $r \leq \dim \mathcal{E}$, for the subspace $\bigwedge_r^k L^*$ to be non trivial, we need to ask $k - r + 1 \leq \operatorname{codim} \mathcal{E}$. Then, under these conditions and for $r \geq 2$,*

$$L \oplus \bigwedge_r^k L^*$$

*is a non-degenerate subspace of $(L \oplus \bigwedge^k L^*, \Omega_L)$ from Example 2.1.1 and, consequently,*

$$\left( L \oplus \bigwedge_r^k L^*, i^*\Omega_L \right)$$

*is a regular multisymplectic vector space, where $i$ is the natural inclusion.*

From now on, we will denote $\Omega_L$ as the multisymplectic form in $L \oplus \bigwedge_r^k L^*$, making abuse of notation.

**Observation 2.1.3.** Notice that for $r > \dim \mathcal{E}$, or $\mathcal{E} = 0$, we recover the canonical multisymplectic vector space $L \oplus \bigwedge^k L$. For simplicity, we will refer to this case as $r = 0$. The only degenerate case is for $r = 1$ and we have

$$\ker \flat_1 = \mathcal{E}.$$

**Remark 2.1.1.** For the sake of clarity in the exposition, we will assume trhoughout the rest of this section the hypotheses that make $(L \oplus \bigwedge_r^k L^*, \Omega_L)$ a regular multisymplectic vector space. More precisely, we will assume $k \leq \dim L$ and, when $r \neq 0$,

- $k - r + 1 \leq \operatorname{codim} \mathcal{E}$;
- $1 < r \leq \dim \mathcal{E}$.

Any further hypotheses will be made explicit in the corresponding results.

**Proposition 2.1.3** ([CIL99]). *Identify both $L$ and $W := \bigwedge_r^k L^*$ a subspace of $L \oplus \bigwedge_r^k L^*$. Then $L$ is $k$-Lagrangian, and $W$ 1-Lagrangian in*

$$\left(L \oplus \bigwedge_r^k L^*, \Omega_L\right).$$

*Proof.* It is clear that $L$ is $k$-isotropic and that $W$ is 1-isotropic. To see that $L$ is $k$-coisotropic, let $(l, \alpha) \in L \oplus \bigwedge_r^k L^*$ such that

$$\Omega_L((l, \alpha), (l_1, 0), \dots, (l_k, 0)) = 0,$$

for every $l_1, \dots, l_k \in L$, that is,

$$\alpha(l_1, \dots, l_k) = 0,$$

for every $l_1, \dots, l_k \in L$. We conclude $\alpha = 0$, and thus,

$$L^{\perp,k} = L = L + \ker \flat_1$$ [2]

Now, to see that $W$ is 1-Lagrangian, let $(l, \alpha) \in L \oplus \bigwedge_r^k L^*$ such that

$$\Omega_L((l, \alpha), (0, \beta_1), (l_2, \beta_2), \dots, (l_k, \beta_k)) = 0,$$

for every $\beta_1, \dots, \beta_k \in \bigwedge_r^k L^*$, and $l_2, \dots, l_k \in L$. Then,

$$\beta_1(l, l_2, \dots, l_k) = 0,$$

for every $l_2, \dots, l_k \in L$. Now we distinguish two cases:

1. Case $r \neq 1$. Then, necessarily $l = 0$, concluding

$$W^{\perp,1} = W = W + \ker \flat_1,$$

because $\ker \flat_1 = 0$.

2. Case $r = 1$. If

$$\beta_1(l, l_2, \dots, l_k) = 0,$$

for every $l_2, \dots, l_k \in L$, we have $l \in \mathcal{E}$ and, therefore,

$$(l, \alpha) \in W + \ker \flat_1,$$

proving that $W$ is 1-Lagrangian.

□

[2] If $r \neq 1$, $\ker \flat_1 = 0$ and, if $r = 1$, $\ker \flat_1 = \mathcal{E}$. In any case, the equality $L = L + \ker \flat_1$ holds.

An important class of multisymplectic vector spaces are those that are multisymplectomorphic to those of Example 2.1.1 and Example 2.1.2. First observe the following:

**Proposition 2.1.4.** *A non-degenerate multisymplectic vector space $(V, \omega)$ is multisymplectomorphic to the one defined in Example 2.1.2 if and only if there exists $\mathcal{E}, L, W \subseteq V$ satisfying:*

- *$L$ is k-Lagrangian with $\mathcal{E} \subseteq L$;*
- *$W$ is 1-Lagrangian and, if $e_1, \dots, e_r \in \mathcal{E}$, we have*

$$\iota_{e_1 \wedge \cdots \wedge e_r} \omega = 0;$$

- *$V = L \oplus W$ and*

$$\dim W = \dim \bigwedge_r^k L^*,$$

  *where the vertical forms are taken with respect to $\mathcal{E}$.*

*Proof.* It is clear the hypothesis imply that the following linear map

$$\phi : W \to \bigwedge_r^k L^*; \ \alpha \mapsto (\iota_\alpha \omega)|_L$$

defines a linear isomorphism. Now, let $\Phi$ be the isomorphism given by

$$\Phi := id_L \oplus \phi : V = L \oplus W \to L \oplus \bigwedge_r^k L^*.$$

We have $\Phi^* \Omega_L = \omega$. Indeed,

$$\begin{aligned}
(\Phi^*\Omega_L)(l_1 + \alpha_1, \dots, l_{k+1} + \alpha_{k+1}) &= \Omega_L((l_1, \phi(\alpha_1), \dots, (l_{k+1}, \phi(\alpha_{k+1}))) = \\
&= \sum_{j=1}^{k+1} (-1)^{j+1} (\phi(\alpha_j))(l_1, \dots, \hat{l}_j, \dots, l_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \omega(\alpha_j, l_1, \dots, \hat{l}_j, \dots, l_{k+1}) \\
&= \sum_{j=1}^{k+1} \omega(l_1, \dots, l_{j-1}, \alpha_j, l_{j+1} \dots, l_{k+1}) = \omega(l_1 + \alpha_1, \dots, l_{k+1} + \alpha_{k+1}),
\end{aligned}$$

proving the result. □

We can prove a weaker version of Proposition 2.1.4. Indeed, given $\mathcal{E}$, $L$, $W$ satisfying the hypotheses, we can canonically identify $\mathcal{E}$ as a subspace of $V/W$ via the isomorphism

$$V/W \cong L.$$

It is easily verified that

$$\iota_{e_1 \wedge \cdots \wedge e_r} \omega = 0,$$

for all $e_1, \dots, e_r \in \mathcal{E}$ is equivalent to

$$\iota_{v_1 \cdots \wedge v_r} \omega = 0,$$

for all $v_1, \dots, v_r \in V$satisfying $\pi(v_i) \in \mathcal{E}$ (identifying $\mathcal{E}$ as a subspace of $V/W$), for every $1 \leq i \leq r$. We have the following:

**Theorem 2.1.1** ([LDS03]). *A non-degenerate multisymplectic vector space $(V, \omega)$ is multisymplectomorphic to $(L \oplus \bigwedge_r^k L^*, \Omega_L)$ if and only if there exists $W \subseteq V$ and $\mathcal{E} \subseteq V/W$ satisfying:*

- *$W$ is 1-Lagrangian and, for all $v_1, \dots, v_r \in V$ with $\pi(v_i) \in \mathcal{E}$ (with $\pi : V \to V/W$ the canonical projection), we have*
$$\iota_{v_1 \wedge \dots \wedge v_r} \omega = 0;$$
- *There is an equality of dimensions*
$$\dim W = \dim \bigwedge_r^k V/W,$$
*where the vertical forms are taken with respect to $\mathcal{E}$.*

To prove it, we will need the following Proposition, which will also be useful in the sequel:

**Proposition 2.1.5.** *Let $(V, \omega)$ be a multisymplectic vector space, and $U$, $W$ be $k$-isotropic, and 1-isotropic subspaces respectively such that*
$$V = U \oplus W.$$
*Then, $U$ is $k$-Lagrangian, and $W$ is 1-Lagrangian.*

*Proof.* We need to prove that
$$U^{\perp,k} = U + \ker \flat_1.$$
Let $u + w \in U^{\perp,k}$, for $u \in U$, $w \in W$. Then, for all $u_1, \dots, u_k \in U$ we have
$$\omega(u + w, u_1, \dots, u_k) = \omega(w, u_1, \dots, u_k) = 0,$$
where we have used that $U$ is $k$-isotropic. We claim that $w \in \ker \flat_1$. Indeed, given $v_i \in V$, for $i = 1, \dots, k$, we can wirte $v_i = u_i + w_i$, with $u_i \in U$, $w_i \in W$. Then,
$$\omega(w, v_1, \dots, v_k) = \omega(w, u_1 + w_1, \dots, u_k + w_k) = \omega(w, u_1, \dots, u_k) = 0,$$
where in the last equality we used that $W$ is 1-isotropic. Therefore, if $u + w \in U^{\perp,k}$, we have
$$u + w \in U + \ker \flat_1,$$
that is
$$U^{\perp,k} \subseteq U + \ker \flat_1,$$
proving that $U$ is $k$-coisotropic and, therefore, $k$-Lagrangian.

To show that $W$ is 1-Lagrangian, let $u + w \in W^{1,\perp}$, with $u \in U$, $w \in W$. Then $u \in \ker \flat_1$. Let $v_i = u_i + +w_i$, $u_i \in U$, $w_i \in W$, $1 \leq i \leq k$. Since $W$ is 1-isotropic, for every $1 \leq i \leq k$
$$\iota_{u \wedge w_i} \omega = \iota_{(u+w) \wedge w_i} \omega = 0.$$

Now, using that $U$ is $k$-isotropic, and $W$ is 1-isotropic,

$$\omega(u, u_1 + w_1, \dots, u_k + w_k) = \sum_{j=1}^{k} \omega(u, u_1, \dots, u_{j-1}, w_j, u_{j+1}, \dots, u_k) = 0.$$

Therefore, $u \in \ker \flat_1$ and $u + w \in W + \ker \flat_1$, showing that

$$W^{\perp,1} = W + \ker \flat_1,$$

ending the proof. □

*Proof (of Theorem 2.1.1).* The proof we give mimics the case $r = 0$ from [SW19]. It is enough to show the existence of a $k$-Lagrangian complement to $W$. By Proposition 2.1.5, we will conclude the proof once we show that there exists a $k$-isotropic complement.

First observe that, since $W$ is 1-Lagrangian, $\iota_\alpha \omega$ induces a form on $V/W$ defining

$$\phi(\alpha)(\pi(v_1), \dots, \pi(v_k)) := \omega(\alpha, v_1, \dots, v_k),$$

for every $\alpha \in W$. This map defines a linear isomorphism

$$\phi : W \to \bigwedge_r^k (V/W)^*,$$

where the vertical forms are taken with respect to $\mathcal{E}$. Take $L$ any complement to $W$ in $V$ and define the linear isomorphism

$$\Phi := id_L \oplus \phi : V = L \oplus W \to L \oplus \bigwedge_r^k (V/W)^*.$$

We will look for subspaces of the form $\Phi^{-1} \circ \mathbf{A}(L)$, where $\mathbf{A} = id_L \oplus A$, with

$$A : L \to \bigwedge_r^k (V/W)^*.$$

For this subspace to be $k$-isotropic, it has to satisfy

$$\omega(\Phi^{-1} \circ \mathbf{A}(l_1), \dots, \Phi^{-1} \circ \mathbf{A}(l_{k+1})) = 0,$$

for all $l_1, \dots, l_{k+1} \in L$. We have

$$\begin{aligned}\omega(\Phi^{-1} \circ \mathbf{A}(l_1), \dots, \Phi^{-1} \circ \mathbf{A}(l_{k+1})) &= \omega(l_1 + \Phi^{-1} A(l_1), \dots, l_{k+1} + \Phi^{-1} A(l_{k+1})) = \\ &\omega(l_1, \dots, l_{k+1}) + \sum_{j=1}^{k+1} (-1)^{j+1} \omega(\Phi^{-1} A(l_j), l_1, \dots, \hat{l}_j, \dots, l_{k+1}) \\ &= \omega(l_1, \dots, l_{k+1}) + \sum_{j=1}^{k+1} (-1)^{j+1} (A(l_j))(\pi(l_1), \dots, \hat{l}_j, \dots, \pi(l_{k+1})).\end{aligned}$$

Notice that the projection $\pi$ restricted to $L$ defines an isomorphism

$$\pi|_L : L \to V/W.$$

Define $A$ closing the following diagram[3]

$$\begin{array}{ccc} L & \xrightarrow{\psi} & \bigwedge_r^k L^* \\ & \searrow^{A} & \uparrow{\scriptstyle \pi^*} \\ & & \bigwedge_r^k (V/W)^* \end{array},$$

where

$$\psi(l) := -\frac{\iota_l \omega}{k+1}.$$

Then,

$$\sum_{j=1}^{k+1} (-1)^{j+1} (A(l_j))(\pi(l_1), \dots, \hat{l}_j, \dots, \pi(l_{k+1})) = \sum_{j=1}^{k+1} (-1)^{j+1} (\pi^* A(l_j))(l_1, \dots, \hat{l}_j, \dots, l_{k+1}) =$$
$$\sum_{j=1}^{k+1} \frac{(-1)^j}{k+1} \omega(l_j, l_1, \dots, \hat{l}_j, \dots, l_{k+1}) = -\omega(l_1, \dots, l_{k+1}),$$

concluding

$$\omega(\Phi^{-1} \circ \mathbf{A}(l_1), \dots, \Phi^{-1} \circ \mathbf{A}(l_{k+1})) = 0,$$

and proving the result. □

This induces the following definition:

**Definition 2.1.5** (Multisymplectic vector space of type $(k, r)$)**.** A **multisymplectic vector space of type** $(k, r)$ is a tuple $(V, \omega, W, \mathcal{E})$ in the hypothesis of Theorem 2.1.1.

In later considerations, the next lemma will result useful:

**Lemma 2.1.1.** *Let $(V, \omega, W, \mathcal{E})$ be a multisymplectic vector space of type $(k, r)$. Then, denoting by $\flat_1$ the induced map*

$$V \xrightarrow{\flat_1} \bigwedge^k V^*,$$

*we have*

$$\bigwedge_{1,r}^k V^* \subseteq \flat_1(V),$$

*where*

$$\bigwedge_{1,r}^k V^* = \bigwedge_1^k V \cap \bigwedge_r^k V^*,$$

*and the vertical forms are taken with respect to $W$ and $\mathcal{E}$[4], respectively.*

[3] Notice that these functions are well defined. Indeed, $\iota_l \omega \in \bigwedge_r^k L^*$ for any $l \in L$.

[4] This latter meaning that $\iota_{e_1 \wedge \cdots \wedge e_r} \alpha = 0$, for every $e_1, \dots, e_r \in V$ with $\pi(e_i) \in \mathcal{E}$, where $\pi : V \to V/W$ is the canonical projection.

*Proof.* By Theorem 2.1.1, it is enough to prove it in the canonical case $V = L \oplus \bigwedge^k L^*, W = \bigwedge^k L^*$. Then, any $k$-form $\alpha \in \bigwedge^k_{1,r} V$ is the pull-back of a $k$-form $\widetilde{\alpha} \in \bigwedge^k_r L^*$. An elementary calculation proves

$$\iota_{\widetilde{\alpha}} \Omega_L = \alpha.$$

□

## 2.2 Multisymplectic manifolds

**Definition 2.2.1** (Multisymplectic manifold)**.** A **multisymplectic manifold** of order $k$ is a pair $(M, \omega)$, where $M$ is a manifold, and $\omega$ is a **multisymplectic form** of order $k$, thats is, a closed $(k+1)$-form. $(M, \omega)$ will be called **non-degenerate** if $\omega_x$ is, for every $x \in M$.

The $j$th-orthogonal defined in Section 2.1 and the notion of $j$-isotropic, $j$-coisotropic, $j$-Lagrangian and regular subspaces generalizes to distributions $\Delta$, and submanifolds $N$, defining it in each subspace $\Delta_x$, or in each tangent space $T_x N$.

**Example 2.2.1.** *We can generalize Example 2.1.1 to manifolds. Fix a manifold L and define*

$$M := \bigwedge^k L,$$

*the bundle of k-forms. We have the tautological k-form*

$$\Theta^k_L|_{\alpha_x}(v_1, \dots, v_k) := \alpha_x(\pi_* v_1, \dots, \pi_* v_k),$$

*where* $\pi : \bigwedge^k L \to L$ *is the natural projection. Define*

$$\Omega^k_L := -d\Theta^k_L.$$

*Then* $(\bigwedge^k L, \Omega^k_L)$ *is a non-degenerate multisymplectic manifold of order k. It is easy to check (see Lemma 3.3.1) that* $\Theta^k_L$ *and* $\Omega^k_L$ *are the only forms on* $\bigwedge^k L$ *satisfying*

$$\alpha^* \Theta^k_L = \alpha, \;\; \alpha^* \Omega^k_L = -d\alpha,$$

*for every k-form (interpreted as a section)* $\alpha : L \to \bigwedge^k L$.

In canonical coordinates $(x^i, p_{i_1,\dots,i_k})$ on $\bigwedge^k L$, we have

$$\Theta^k_L = p_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

and

$$\Omega^k_L = -dp_{i_1,\dots,i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

This immediately shows that the vertical distribution $W^k_L$ associated to the vector bundle $\bigwedge^k L \to L$ is 1-isotropic. Additionally, we have:

**Proposition 2.2.1.** *$W_L^k$ defines a 1-Lagrangian distribution. Furthermore, a form (interpreted as a section)*

$$\alpha : L \to \bigwedge^k L$$

*defines a k-Lagrangian submanifold if and only if it is closed.*

*Proof.* By Proposition 2.1.5, it is enough to show that $\alpha$ defining a $k$-isotropic submanifold is equivalent to $\alpha$ being closed (this would imply that $\alpha(L)$ is $k$-Lagrangian, and that $W_L^k$ is 1-Lagrangian, since they are complementary). Indeed, by Proposition 2.1.2, $\alpha(L)$ is $k$-isotropic if and only if

$$0 = \alpha^* \Omega_L^k = -d\alpha,$$

that is, if and only if $\alpha$ is closed. □

There is another relevant example that generalizes Example 2.1.2:

**Example 2.2.2.** *Let L be a manifold and $\mathcal{E}$ be a regular distribution, where $r, k, \mathcal{E}_x, T_xL$ are in the hypotheses of Remark 2.1.1 for every $x \in L$. Define*

$$\bigwedge_r^k L := \left\{ \alpha_x \in \bigwedge^k T_x^*L \ : \ \iota_{e_1 \wedge \cdots \wedge e_r} \alpha_x = 0, \ \forall e_1, \dots, e_r \in \mathcal{E}_x \right\}.$$

*It is easy to check that $\bigwedge_r^k L$ defines a non-singular submanifold of $\bigwedge^k L$. Therefore,*

$$\left( \bigwedge_r^k L, \Omega_L \right)$$

*is a multisymplectic manifold of order k.*

**Remark 2.2.1.** Just like in Section 2.1, throughout the rest of the text we will assume the conditions that make $\bigwedge_r^k L$ a regular multisymplectic manifold.

A natural question to ask is what are the necessary (and sufficient) conditions for a multisymplectic manifold $(M, \omega)$ to be locally multisymplectomorphic to either of the models presented in Example 2.2.1 or in Example 2.2.2. Of course, if it were the case, the multisymplectic vector space $(T_xM, \omega_x)$ would necessarily be of type $(k, r)$ (for the corresponging values in the model).

**Definition 2.2.2** ([LDS03] Multisymplectic manifold of type $(k, r)$)**.** A multisymplectic manifold of type $(k, r)$ is a tuple $(M, \omega, W, \mathcal{E})$, where $(T_xM, \omega_x, W_x, \mathcal{E}_x)$ is a multisymplectic vector space of type $(k, r)$ and $W$ is a regular integrable distribution.

In [Mar88], G. Martin gave the characterization for multisymplectic manifolds of type $(k, 0)$.

**Theorem 2.2.1** ([Mar88] Darboux theorem for multisymplectic manifolds of type $(k, 0)$)**.** *Let* $(M, \omega, W)$ *be a multisymplectic manifold of type* $(k, 0)$*. Then, around each point* $x \in M$ *there exists a neighborhood* $U$ *of* $x$ *in* $M$*, a manifold* $L$*, and a multisymplectomorphism*

$$\phi : (U, \omega) \to (V, \Omega_L)$$

*where* $V$ *is an open subset of* $\bigwedge^k L$*.*

And, in [LDS03], M. de León et. al. generalized the result to multisymplectic manifolds of type $(k, r)$. We omit the proof.

**Theorem 2.2.2** ([LDS03] Darboux theorem for multisymplectic manifolds of type $(k, r)$)**.** *Let* $(M, \omega, W, \mathcal{E})$ *be a multisymplectic manifold of type* $(k, r)$*. Then, around each point* $x \in M$ *there exists a neighborhood* $U$ *of* $x$ *in* $M$*, a manifold* $L$*, and a multisymplectomorphism*

$$\phi : (U, \omega) \to (V, \Omega_L)$$

*where* $V$ *is an open subset of* $\bigwedge^k_r L$*.*

# Chapter 3

# Hamiltonian structures on multisymplectic manifolds

In this chapter we generalize the interpretation of dynamics as a Lagrangian submanifold to multisymplectic geometry. First, we define the Poisson bracket of Hamiltonian multivector fields in Section 3.1 (a generalization of the Poisson bracket) in order to give the main theorem in Section 3.2, which will consist of endowing the vector bundle of multivectors $\bigvee_q M$ with a multisymplectic structure. A different definition of this multisymplectic structure is studied in Section 3.3.

## 3.1 Hamiltonian multivector fields and forms

**Definition 3.1.1** ([CIL96] Hamiltonian multivector field, Hamiltonian form)**.** Let $(M, \omega)$ be a multisymplectic manifold of order $k$. A multivector field

$$U : M \to \bigvee_q M$$

will be called a **Hamiltonian multivector field** if there exists a $(k-q)$-form on $M$, $\alpha$, such that

$$\iota_U \omega = d\alpha.$$

In this context, $\alpha$ is called the **Hamiltonian form** associated to $U$. Furthermore, $U$ will be called a **locally Hamiltonian multivector field** if $\iota_U\omega$ is closed. Of course, if $U$ is Hamiltonian, it is locally Hamiltonian.

We will denote by $\mathfrak{X}^q_H(M)$ the space of all Hamiltonian multivector fields of order $q$, and by $\Omega^l_H(M)$ the space of all Hamiltonian $l$-forms.

There is certain "correspondence" between Hamiltonian multivector fields and Hamiltonian forms. However, this correspondance is not well defined, a Hamiltonian multivector field $U$ can be associated to different Hamiltonian forms, and viceversa. Nevertheless, if

$$\iota_U \omega = d\alpha = d\beta,$$

for some $\alpha, \beta \in \Omega^l_H(\Omega)$, we have that

$$d(\alpha - \beta) = 0.$$

Therefore, we obtain a well defined epimorphism

$$\mathfrak{X}^q_H(M) \xrightarrow{\flat_q} \Omega^{k-q}_H(M)/Z^{k-q}(M) =: \widehat{\Omega}^{k-q}_H(M),$$

where $Z^{k-q}(M)$ is the space of all closed forms, mapping each Hamiltonian multivector field $U$ to the class of Hamiltonian forms $[\alpha]$ satisfying

$$\iota_U \omega = d\alpha.$$

We would like this map to be inyective, and we can achieve this by quotienting $\mathfrak{X}^q_H(M)$ by $\ker \flat_q$, which is the space of all multivector fields $U$ satisfying

$$\iota_U \omega = 0.$$

Therefore, defining

$$\widehat{\mathfrak{X}}^q_H(M) := \mathfrak{X}^q_H(M)/\ker \flat_q,$$

we obtain isomorphisms between the spaces

$$\begin{array}{ccccc} \widehat{\mathfrak{X}}^1_H(M) & \cdots & \widehat{\mathfrak{X}}^q_H(M) & \cdots & \widehat{\mathfrak{X}}^k_H(M) \\ \big\updownarrow{\scriptstyle \flat_1} & & \big\updownarrow{\scriptstyle \flat_q} & & \big\updownarrow{\scriptstyle \flat_k} \\ \widehat{\Omega}^{k-1}_H(M) & \cdots & \widehat{\Omega}^{k-q}_H(M) & \cdots & \widehat{\Omega}^0_H(M) \end{array}.$$

Of course, these isomorphisms induce an isomorphism between the corresponding graded vector spaces

$$\widehat{\mathfrak{X}}_H(M) := \bigoplus_{q=1}^{k} \widehat{\mathfrak{X}}^q_H(M) \xrightarrow{\flat} \widehat{\Omega}_H(M) := \bigoplus_{q=1}^{k} \widehat{\Omega}^{k-q}_H(M).$$

We can try to endow these spaces with a graded Lie algebra structure. Given the isomorphism, it would be enough to define the bracket in one of the spaces and obtain the induced bracket in the other via $\flat$.

**Proposition 3.1.1** ([CIL96]). *Let $(M, \omega)$ be a multisymplectic manifold, and $U, V$ be Hamiltonian multivector fields of degree $p, q$, respectively. Then, $[U, V]$ is a Hamiltonian multivector field of degree $p + q - 1$, where $[\cdot, \cdot]$ denotes the Schouten-Nijenhuis bracket (see [Vai94]).*

*Proof.* We have the equality (see [Vai94])

$$\iota_{[U,V]}\omega = -d\iota_{U\wedge V}\omega,$$

which proves the proposition. □

Given the equality
$$\iota_{[U,V]}\omega = -d\iota_{U\wedge V}\omega$$
from Proposition 3.1.1, we have that whenever $U \in \ker \flat_p$ (or $V \in \ker \flat_q$),
$$[U,V] \in \ker \flat_{p+q-1}.$$
Therefore, we obtain a well defined bracket
$$\widehat{\mathfrak{X}}^p_H(M) \times \widehat{\mathfrak{X}}^q_H(M) \to \widehat{\mathfrak{X}}^{p+q-1}_H(M);\ (\widehat{U},\widehat{V}) \mapsto [\widehat{U},\widehat{V}] := \widehat{[U,V]},$$
where $\widehat{U}$ denotes the class of $U$ modulo $\ker \flat_q$. By the previous considerations, we define the induced bracket in $\widehat{\Omega}_H(M)$ through the following commutative diagram,
$$\begin{array}{ccc} \widehat{\Omega}^l_H(M) \times \widehat{\Omega}^m_H(M) & \xrightarrow{\{\cdot,\cdot\}} & \widehat{\Omega}^{1+l+m-k}_H(M) \\ \updownarrow{\scriptstyle \flat_{k-l}\times\flat_{k-m}} & & \updownarrow{\scriptstyle \flat_{2k-l-m-1}} \\ \widehat{\mathfrak{X}}^{k-l}_H(M) \times \widehat{\mathfrak{X}}^{k-m}_H(M) & \xrightarrow{[\cdot,\cdot]} & \widehat{\mathfrak{X}}^{2k-l-m-1}_H(M) \end{array}.$$
This bracket is given by
$$\{\hat{\alpha},\hat{\beta}\} = -\widehat{\iota_{U\wedge V}\omega},$$
where $\iota_U\omega = d\alpha$, $\iota_V\omega = d\beta$, and satisfies the following equalities (which follow easily from the equalities of Schouten-Nijenhuis bracket [Vai94])

*i)*
$$\{\widehat{\alpha},\widehat{\beta}\} = (-1)^{l_1 l_2}\{\widehat{\beta},\widehat{\alpha}\};$$

*ii)*
$$(-1)^{l_1(l_3-1)}\{\widehat{\alpha},\{\widehat{\beta},\widehat{\gamma}\}\} + (-1)^{l_2(l_1-1)}\{\widehat{\beta},\{\widehat{\gamma},\widehat{\alpha}\}\} + (-1)^{l_3(l_2-1)}\{\widehat{\gamma},\{\widehat{\alpha},\widehat{\beta}\}\} = 0,$$

for $\widehat{\alpha} \in \widehat{\Omega}^{l_1}_H(M), \widehat{\beta} \in \widehat{\Omega}^{l_2}_H(M), \widehat{\gamma} \in \widehat{\Omega}^{l_3}_H(M)$. However, this bracket does not define a graded Lie algebra and we need to modify the definition slightly to get a bracket that does. First, recall that a graded Lie bracket needs to satisfy
$$\deg\{\widehat{\alpha},\widehat{\beta}\} = \deg\widehat{\alpha} + \deg\widehat{\beta},$$
for certain notion of degree. Now, since the subspace $\widehat{\Omega}^{k-1}_H(M)$ is closed under $\{\cdot,\cdot\}$, we are forced to set
$$\deg\widehat{\alpha} := 0,$$
for $\alpha \in \widehat{\Omega}^{k-1}_H(M)$. Therefore, one is tempted to define
$$\deg\widehat{\alpha} := k-1-(\text{order of } \alpha),$$
for $\widehat{\alpha} \in \widehat{\Omega}_H(M)$. And, indeed, for $\widehat{\alpha} \in \widehat{\Omega}^l_H(M), \widehat{\beta} \in \widehat{\Omega}^m_H(M)$, we have
$$\deg\{\widehat{\alpha},\widehat{\beta}\} = k-1-(1+l+m-k) = 2k-l-m-2 = (k-1-l)+(k-1-m) = \deg\widehat{\alpha}+\deg\widehat{\beta}.$$
We can now define
$$\{\widehat{\alpha},\widehat{\beta}\}^{\bullet} := (-1)^{\deg\widehat{\alpha}}\{\widehat{\alpha},\widehat{\beta}\},$$
and we have that

*i)*

$$\{\widehat{\alpha}, \widehat{\beta}\}^{\bullet} = -(-1)^{\deg \widehat{\alpha} \deg \widehat{\beta}} \{\widehat{\beta}, \widehat{\alpha}\}^{\bullet},$$

*ii)*

$$(-1)^{\deg \widehat{\alpha} \deg \widehat{\gamma}} \{\widehat{\alpha}, \{\widehat{\beta}, \widehat{\gamma}\}^{\bullet}\}^{\bullet} + \text{cycl.} = 0.$$

Summarizing, we have proved

**Theorem 3.1.1** ([CIL96]). $(\widehat{\Omega}_H(M), \{\cdot,\cdot\}^{\bullet})$ *is a graded Lie algebra.*

**Remark 3.1.1.** Of course, restricting this structure to the forms of order $k-1$ we obtain the Lie algebra $(\widehat{\Omega}^{k-1}_H(M), \{\cdot,\cdot\}^{\bullet})$. This Lie algebra is of particular importance in the study of multisymplectic manifolds, since $(k-1)$-forms represent the *conserved quantities* and *currents* of classical field theory and calculus of variations.

Similar to the characterization of coisotropic submanifold of a symplectc manifold in terms of the Poisson algebra, we can prove the following result.

**Proposition 3.1.2.** *Let $i : N \hookrightarrow M$ be a $k$-coisotropic submanifold. Then*

$$\widehat{I}_N := \{\widehat{\alpha} \in \widehat{\Omega}^{k-1}_H(M) : \; i^*\alpha = 0\}^{1}$$

*defines a subalgebra of $(\widehat{\Omega}^{k-1}_H(M), \{\cdot,\cdot\}^{\bullet})$.*

*Proof.* Let $\widehat{\alpha}, \widehat{\beta} \in \widehat{I}_N$. Then, there are vector fields $X_\alpha, X_\beta$ satisfying

$$\iota_{X_\alpha}\omega = d\alpha, \;\; \iota_{X_\beta}\omega = d\beta.$$

Since $i^*\alpha, i^*\beta = 0$, we conclude that $X_\alpha, X_\beta$ take values in $(TN)^{\perp,k} \subseteq TN + \ker \flat_1$. Without loss of generality, we can assume that $X_\alpha, X_\beta$ take values in $TN$. Now, since

$$\{\widehat{\alpha}, \widehat{\beta}\}^{\bullet} = (-1)^{(k-1)} \widehat{\iota_{X_\alpha \wedge X_\beta}\omega},$$

and $X_\alpha, X_\beta$ take values in $(TN)^{\perp,k}$ and $TN$ , we have

$$i^*\iota_{X_\alpha \wedge X_\beta}\omega = 0,$$

concluding that

$$\{\widehat{\alpha}, \widehat{\beta}\}^{\bullet} \in \widehat{I}_N.$$

□

**Remark 3.1.2.** When $(M, \omega)$ is non-degenerate, each Hamiltonian $(k-1)$-form $\alpha$ defines an *unique* vector field $X_\alpha$ satisfying

$$\iota_{X_\alpha}\omega = d\alpha.$$

[1]That is, the space of all Hamiltonian $(k-1)$-forms that have a representative in $\ker i^*$.

Therefore, the bracket

$$\{\alpha, \beta\} = \iota_{X_\alpha \wedge X_\beta}\omega$$

is well defined. This, however, does not define a Lie algebra since Jacobi identity holds up to a closed form. Nevertheless, it does defines an algebraic structure called an $L_\infty$-algebra. Proposition 3.1.2 is also true in this context, that is, to each coisotropic submanifold $N$, there is the corresponding $L_\infty$-algebra of forms that are zero on $N$.

Let us now briefly discuss conserved quantities. Consider a locally decomposable Hamiltonian multivector field of order $q$,

$$\iota_{X_H}\omega = dH,$$

where $H \in \Omega^{k-q}(M)$ is the Hamiltonian. We will consider as a solution any immersion $\phi : \Sigma \to M$, where $\dim \Sigma = q$, satisfying

$$\phi_* U = X_H,$$

where $U$ is some nowhere vanishing multivector field of order $q$ on $\Sigma$. Then, a conserved quantity (for the solution $\phi$) is a $(q-1)$-form satisfying

$$d\phi^*\alpha = 0.$$

In terms of possibly non-decomposable (nor integrable) multivector fields, this notion extends as follows

**Definition 3.1.2** (Conserved quantity)**.** A conserved quantity for a Hamiltonian multivector field $X_H \in \mathfrak{X}^q(M)$ is a $(q-1)-$form $\alpha$ on $M$ satisfying

$$\langle d\alpha, X_H \rangle = 0.$$

Then, for Hamiltonian forms, we have the following

**Proposition 3.1.3.** *Let $X_H$ be a Hamiltonian multivector field of order $q$, with Hamiltonian form $H \in \Omega^{k-q}(M)$ and $\alpha$ be a Hamiltonian form of order $q-1$. Then $\alpha$ is a conserved quantity for $X_H$ if and only if*

$$\{\widehat{\alpha}, \widehat{H}\}^{\bullet} = 0.$$

For a treatment of conserved quantities and moment maps using the $L_\infty-$algebra strcuture of observables, we refer to [RW15; RWZ20].

## 3.2 Hamiltonian multivector fields as Lagrangian submanifolds

Given a symplectic manifold $(M, \omega)$, we can endow its tangent bundle with a symplectic structure using the bundle ismorphism

$$TM \xrightarrow{\flat} T^*M,$$

and the canonical symplectic form in $T^*M$. With this definition and interpreting a vector field $X : M \to TM$ as a submanifold, $X$ is $(1-)$Lagrangian if and only if it is locally Hamiltonian.

We would like to generalize this result to general multisymplectic manifolds and multivector fields of aribitrary order $q$

$$U : M \to \bigvee_q M.$$

In [CIL96], the authors prove a generalization of the result to vector fields in multisymplectic manifolds

$$X : M \to TM,$$

endowing the tangent bundle $TM$ with a multisymplectic structure via the complete lift of forms. We will explore how to generalize this method in *Section* 3.3. In the meantime, let us begin by defining a multisymplectic structure on $\bigvee_q M$.

Given a multisymplectic manifold $(M, \omega)$ of order $k$, we have the induced map by contraction

$$\bigvee_q M \xrightarrow{\flat_q} \bigwedge^{k+1-q} M;\ u \mapsto \iota_u \omega.$$

Using the canonical multisymplectic form $\Omega_M^{k+1-q}$ on $\bigwedge^{k+1-q} M$, we can define the closed form (in fact, exact)

$$\widetilde{\Omega}_M^q := (\flat_q)^* \Omega_M^{k+1-q},$$

which endows $\bigvee_q M$ with a multisymplectic structure of order $(k+1-q)$. Notice that, for $q = 1$, the order of the multisymplectic structure on $TM$ is the order of the multisymplectic structure on $M$.

**Remark 3.2.1.** Even if $\omega$ is non-degenerate, $\widetilde{\Omega}_M^q$ could have non trivial kernel. This motivates the study of "general" multisymplectic structures that we have adopted in the text, which provides a way of interpreting multivector fields as Lagrangian submanifolds of (possible degenerate) multisymplectic manifolds.

Denote by $\widetilde{W}_M^q$ the vertical distribution associated to the vector bundle

$$\bigvee_q M \to M.$$

Since $\flat_q$ is a bundle map

$$\begin{array}{ccc} \bigvee_q M & \xrightarrow{\flat_q} & \bigwedge^{k+1-q} M \\ & \searrow \quad \swarrow & \\ & M & \end{array},$$

we have that

$$(\flat_q)_* \widetilde{W}_M^q \subseteq W_M^{k+1-q},$$

where $W_M^{k+1-q}$ is the vertical distribution of the vector bundle

$$\bigwedge^{k+1-q} M \to M.$$

Now, recall that $W_M^{k+1-q}$ defines a 1-Lagrangian distribution. Therefore, we have

**Proposition 3.2.1.** *$\widetilde{W}_M^q$ defines a 1-isotropic distribution on $(\bigvee_q M, \widetilde{\Omega}_M^q)$.*

Now we can prove the main result of this section.

**Theorem 3.2.1.** *Let $(M, \omega)$ be a multisymplectic manifold of order $k$. Then, a multivector field*

$$U : M \to \bigvee_q M$$

*is locally Hamiltonian if and only if it defines a $(k+1-q)$-Lagrangian submanifold in $(\bigvee_q M, \widetilde{\Omega}_M^q)$.*

*Proof.* With Proposition 2.1.5 in mind, since $\widetilde{W}_M^q$ is 1-isotropic by Proposition 3.2.1, and we have the decomposition

$$T\bigvee_q M\Big|_{U(M)} = U_*(TM) \oplus \widetilde{W}_M^q\Big|_{U(M)},$$

we only need to check wether $U$ defines a $(k+1-q)$-isotropic submanifold or, equivalently, wether

$$U^*\widetilde{\Omega}_M^q = 0.$$

$$\begin{array}{ccc} \bigvee_q M & \xrightarrow{\ \flat_q\ } & \bigwedge^{k+1-q} M \\ {\scriptstyle U}\uparrow & \nearrow_{\flat_q(U)=\iota_U\omega} & \\ M & & \end{array}$$

We have that

$$\begin{aligned} U^*\widetilde{\Omega}_M^q &= U^*\flat_q^*\Omega_M^{k+1-q} = (\flat_q \circ U)^*\Omega_M^{k+1-q} \\ &= (\iota_U\omega)^*\Omega_M^{k+1-q} = -d\iota_U\omega, \end{aligned}$$

where in the last equality we have used that $\alpha^*\Omega_Q^k = -d\alpha$, for any form $\alpha : Q \to \bigwedge^k Q$. We conclude that $U$ is $k$-Lagrangian if and only if

$$0 = U^*\widetilde{\Omega}_M^q = -d\iota_U\omega,$$

that is, if and only if $U$ is locally Hamiltonian. □

## 3.3 Complete lift of forms to multivector bundles

In [CIL99], the authors prove that $(TM, \omega^c)$ is a non-degenerate multisymplectic manifold when $\omega$ is a non-degenerate multisymplectic form on $M$. Here $\omega^c$ denotes the complete lift of the form. We would like to generalize this procedure to arbitrary multivector bundles

$$\bigvee_q M.$$

Let us begin by recalling that $\omega^c$ is the unique $(k+1)$-form on $TM$ satisfying

$$X^*\omega^c = \pounds_X \omega,$$

for every vector field

$$X : M \to TM.$$

Recalling the Cartan formula

$$\pounds_X \omega = d\iota_X \omega + \iota_X d\omega,$$

we define the Lie derivative of a $\omega$ with respect to a multivector field

$$U : M \to \bigvee_q M$$

as the $(k+2-q)$-form

$$\pounds_U \omega := \iota_U d\omega + (-1)^{q+1} d\iota_U \omega.$$

**Theorem 3.3.1** (Definition of complete lift)**.** *Given a manifold $M$, and $\omega \in \Omega^{k+1}(M)$, there exists an unique $(k+2-q)$-form on $\bigvee_q M$, $\omega_q^c$, such that*

$$U^*\omega_q^c = \pounds_U \omega,$$

*for every multivector field*

$$U : M \to \bigvee_q M.$$

To prove uniqueness, it suffies to study the linear problem.

**Lemma 3.3.1.** *Let $X, Y$ be vector spaces and $\pi : Y \to X$ be an epimorphism. Then, if $k+1 \leq \dim X$, a form $\omega \in \bigwedge^{k+1} Y^*$ is characterized by the pull-backs of all sections*

$$\phi : X \to Y.$$

*That is, if there is another $(k+1)$-form $\alpha$ on $Y$ such that $\phi^*\alpha = \phi^*\omega$, for every section $\phi : X \to Y$ of $\pi$, then*

$$\alpha = \omega.$$

*Proof.* It is clear that $\omega$ is characterized by the induced linear map

$$\omega : \bigwedge^{k+1} Y \to \mathbb{R},$$

and that, if $\phi^*\alpha = \phi^*\omega$, for certain form $\alpha \in \bigwedge^{k+1} Y^*$, the following diagram commutes.

$$\begin{array}{ccc} \bigwedge^{k+1} Y & \xrightarrow{\omega} & \mathbb{R} \\ {\scriptstyle \phi_*} \uparrow \downarrow {\scriptstyle \pi_*} & \nearrow_{\phi^*\alpha} & \\ \bigwedge^{k+1} X & & \end{array} .$$

Therefore, if we can prove that

$$\bigwedge^{k+1} Y = \left\langle \phi_* \left( \bigwedge^{k+1} X \right),\ \phi : X \to Y \text{ section} \right\rangle,$$

we would have $\omega = \alpha$, since they would coincide in a set of generators. Identify $X$ as a subspace of $Y$. We have

$$\bigwedge^{k+1} Y = \bigwedge^{k+1} (X \oplus \ker \pi) = \bigoplus_{l=0}^{k+1} \left( \bigwedge^{l} X \wedge \bigwedge^{k+1-l} \ker \pi \right).$$

We will prove that

$$\bigwedge^{l} X \wedge \bigwedge^{k+1-l} \ker \pi \subseteq \left\langle \phi_* \left( \bigwedge^{k+1} X \right),\ \phi : X \to Y \text{ section} \right\rangle.$$

Let

$$x_1 \wedge \cdots \wedge x_l \wedge y_{l+1} \wedge \cdots \wedge y_{k+1} \in \bigwedge^{l} X \wedge \bigwedge^{k+1-l} \ker \pi,$$

where $x_i \in X, y_j \in \ker \pi$ are linearly independent vectors. Extend $x_1, \dots, x_{k+1-l}$ to $k+1$ linearly independent vectors on $X$ (here we are using $\dim X \geq k+1$),

$$x_1, \dots, x_{k+1}$$

and take a section $\phi : X \to Y$ such that

$$\phi(x_i) = x_i, i = 1, \dots, k,\ \ \phi(x_{k+1}) = x_{k+1} + y_{k+1}.$$

Then

$$\begin{aligned} & x_1 \wedge \cdots \wedge x_k \wedge y_{k+1} = \\ & \phi_*(x_1 \wedge \cdots \wedge x_{k+1}) - x_1 \wedge \cdots \wedge x_{k+1} \in \left\langle \phi_* \left( \bigwedge^{k+1} X \right),\ \phi : X \to Y \text{ section} \right\rangle. \end{aligned}$$

With a similar argument we can show that

$$x_1 \wedge \cdots \wedge x_{k-1} \wedge y_k \wedge x_{k+1} \in \left\langle \phi_* \left( \bigwedge^{k+1} X \right),\ \phi : X \to Y \text{ section} \right\rangle.$$

Now, defining another section (which we name the same making abuse of notation) $\phi$ satisfying

$$\phi(x_i) = x_i,\ i = 1, \dots, k-1,\ \ \phi(x_k) = x_k + y_k, \phi(x_{k+1}) = x_{k+1} + y_{k+1},$$

we have

$$\phi_*(x_1 \wedge \cdots \wedge x_{k+1}) = x_1 \wedge \cdots \wedge x_{k-1} \wedge (x_k + y_k) \wedge (x_{k+1} + y_{k+1})$$

which, by the previous considerations implies

$$x_1 \wedge \cdots \wedge x_{k-1} \wedge y_k \wedge y_{k+1} \in \left\langle \phi_* \left( \bigwedge^{k+1} X \right),\ \phi : X \to Y \text{ section} \right\rangle.$$

Now, iterating this argument we conclude

$$x_1 \wedge \cdots \wedge x_l \wedge y_{l+1} \wedge \cdots \wedge y_{k+1} \in \left\langle \phi_* \left( \bigwedge^{k+1} X \right),\ \phi : X \to Y \text{ section} \right\rangle,$$

proving the result. □

*Proof of Theorem 3.3.1.* By Lemma 3.3.1, if we find a form $\omega_q^c$ on $\bigvee_q M$ satisfying

$$U^* \omega_q^c = \pounds_U \omega,$$

the result would follow. Consider the induced maps by $\omega$ and $d\omega$ on $\bigvee_q M$,

$$\bigvee_q M \xrightarrow{\widetilde{\flat}_q := \iota_\bullet d\omega} \bigwedge^{k+2-q} M, \qquad \bigvee_q M \xrightarrow{\flat_q := \iota_\bullet \omega} \bigwedge^{k+1-q} M,$$

and define a $(k+2-q)$-form on $\bigvee_q M$ by

$$\omega_q^c := (\widetilde{\flat}_q)^* \Theta_M^{k+2-q} + (-1)^q (\flat_q)^* \Omega_M^{k+1-q}.$$

Then, by definition of $\Theta_M^{k+2-q}$, and $\Omega_M^{k+1-q}$ we have that for all multivector fields $U : M \to \bigvee_q M$,

$$U^* \omega_q^c = \iota_U d\omega + (-1)^{q+1} d\iota_U \omega = \pounds_U \omega,$$

finishing the proof. □

**Remark 3.3.1.** Now that we have generalized the complete lift of forms to arbitrary multivector bundles, given a multisymplectic manifold $(M,\omega)$ we have two ways of inducing a multisymplectic structure on $\bigvee_q M$, the one constructed in Section 3.2, and the complete lift from Theorem 3.3.1. However, because $\omega$ is closed, the map $\widetilde{\flat}_q$ of the proof of Theorem 3.3.1 is trivial and thus,

$$\omega_q^c = (-1)^q(\flat_q)^*\Omega_M^{k+1-q} = (-1)^q\widetilde{\Omega}_M^q$$

and we conclude that, up to sign, both multisymplectic structures are equal.

# Chapter 4

# Coisotropic submanifolds

In this chapter we study in detail coisotropic subamanifolds, with particular atention to coisotropic submanifolds of $\bigwedge_r^k L$.

In Section 4.1 we give a local form for certain type of coisotropic submanifolds and in Section 4.2 we prove a coisotropic reduction theorem for this kind of submanifolds.

## 4.1 Local form of coisotropic submanifolds

Weinstein gave the first normal form[1] theorem for Lagrangian submanifolds in the context of symplectic geometry.

**Theorem 4.1.1** ([Wei71] Weinstein's Lagrangian neighborhood Theorem)**.** *Let $(M,\omega)$ be a symplectic manifold and $L \hookrightarrow M$ be a Lagrangian submanifold. Then there are neighborhoods $U$, $V$ of $L$ in $M$, and in $T^*L$ (identifying $L$ with the zero section) respectively, and a symplectomorphism*

$$\phi : U \to V.$$

This result has been generalized to multisymplectic manifolds of type $(k,0)$ by G. Martin [Mar88], and extended to multiysmplectic manifolds of type $(k,r)$ by M. de Leon *et al.* [LDS03].

**Theorem 4.1.2** ([LDS03])**.** *Let $(M,\omega,W,\mathcal{E})$ be a multisymplectic manifold of type $(k,r)$, and $L \hookrightarrow M$ be a $k$-Lagrangian submanifold complementary to $W$, that is, such that*

$$TL \oplus W\big|_L = TM\big|_L.$$

*Then there are neighborhoods $U$, $V$ of $L$ in $M$, and of $L$ in $\bigwedge_r^k L$ (identifying $L$ as the zero section), where the horizontal forms are taken with respect to $\mathcal{E}$ under the identification*

$$TL = TM/W,$$

[1] During the text, we reserve the term *normal form* for a classification of a neighborhood of an entire submanifold (like in Theorem 4.1.1, Theorem 4.1.2), and we use the term *local form* for a classification of a neighborhood around any point of a submanifold (like in Theorem 4.1.4).

*and a multisymplectomorphism*

$$\psi : U \to V,$$

*which is the identity on L and satisfies*

$$\psi_* W = W_L^k,$$

*where $W_L^k$ denotes the vertical distribution on $\bigwedge_r^k L^*$.*

*Proof.* Define the vector bundle isomorphism

$$\phi : W|_L \to \bigwedge_r^k L;\ \phi(w_l) := (\iota_{w_l}\omega)|_L.$$

By the tubular neighborhood theorem, we may identify a neighborhood $U$ of $L$ in $W|_L$ with a neighborhood of $L$ in $M$. Under the previous identificaction, let $V := \phi(U)$ and define

$$\widetilde{\omega} := \phi_* \omega.$$

Following the same reasoning as in Proposition 2.1.4, we have $\widetilde{\omega} = \Omega_L^k$ on $L$. Furthermore, since $\phi$ is a vector bundle isomorphism, $\phi$ preserves fibers and we have $\phi_* W|_U = (W_L^k)|_V$. This implies that $W_L^k$ not only defines a 1-isotropic distribution for $\Omega_L^k$, but also for $\widetilde{\omega}$. To build the multisymplectomorphism $\psi$, we will make use of *Moser's trick* with the family of forms

$$\Omega_t := (1-t)\Omega_L^k + t\widetilde{\omega}.$$

More precisely, we will look for a time dependent vector field $X_t$ on $V$ such that its flow $\phi_t$ satisfies

$$\phi_t^* \Omega_t = \Omega_L^k,$$

for every $t$. To achieve this, it will be sufficient to look for a time dependent vector field $X_t$ such that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_t^* \Omega_t\right) = \pounds_{X_t}\Omega_t + \frac{\mathrm{d}\Omega_t}{\mathrm{d}t} = d\iota_{X_t}\Omega_t + \widetilde{\omega} - \Omega_L^k.$$

Now, if we denote by $\pi_t$ multiplication by $t$ in $\bigwedge_r^k L$, by reducing neighborhoods if necessary, we get a well defined map

$$\pi_t : V \to V,$$

for $0 \leq t \leq 1$. By the relative Poincaré Lemma, we have

$$\widetilde{\omega} = d\left(\int_0^1 \pi_t^* \iota_\Delta \widetilde{\omega} dt\right),$$

where $\Delta$ is the dilation vector field. Therefore, if we define

$$\widetilde{\theta} := -\int_0^1 \pi_t^* \iota_\Delta \widetilde{\omega} dt,$$

it follows that $\widetilde{\omega} = -d\widetilde{\theta}$, where $\widetilde{\theta} = 0$ on $L$ (because $\Delta = 0$ on $L$). Since we need $\Omega_L^k - \widetilde{\omega} = -d(\Theta_L^k - \widetilde{\theta}) = d\iota_{X_t}\Omega_t$, it will be enough to look for $X_t$ satisfying

$$\iota_{X_t}\Omega_t = \widetilde{\theta} - \Theta_L^k.$$

Recall that $\widetilde{\omega} = \Omega_L^k$ on $L$ and, therefore $\Omega_t = \Omega_L^k$ on $L$. Since this form is nondegenerate, by reducing the neighborhoods further, we can assume that $\Omega_t$ is nondegenerate on $V$, for every $t \in [0,1]$. Notice that $\iota_Y\left(\widetilde{\theta} - \Theta_L^k\right) = 0$, for any vector field $Y$ that takes values in $W_L^k$, and that

$$\iota_{E_1\wedge\cdots\wedge E_r}\widetilde{\theta} = \iota_{E_1\wedge\cdots\wedge E_r}\Theta_L^k = 0,$$

for vector fields $E_1, \dots, E_r$ such that $\pi(E_i)$ takes values in $\mathcal{E} \subset L$ (where $\pi : \bigwedge_r^k L \to L$ is the canonical projection). These last two properties, together with Lemma 2.1.1, imply that there exists an unique time-dependent vector field $X_t$ with values in $W_L^k$ satisfying

$$\iota_{X_t}\Omega_t = \widetilde{\theta} - \Theta_L^k.$$

Furthermore, since $\widetilde{\theta} = \Theta_L^k = 0$ on $L$, $X_t = 0$ on $L$, and its flow is globally defined on $L$. It follows that we can assume that $\phi_t$ (the flow of $X_t$) is defined on $V$ for $0 \leq t \leq 1$ by reducing the neighborhoods further. Finally, for $t = 1$, this flow satisfies

$$\phi_1^*\widetilde{\omega} = \Omega$$

and preserves fibers, because $X_t$ takes values in $W_L^k$. Defining

$$\psi := (\phi_1)^{-1} \circ \phi,$$

we get the desired multisymplectomorphism. □

We can use Theorem 4.1.2 to give a local form for vertical $k$-coisotropic submanifolds $N \hookrightarrow M$ of a multisymplectic manifold of type $(k, r)$, where vertical means that

$$W\big|_N \subseteq TN.$$

**Theorem 4.1.3** (Local form of $k$-coisotropic submanifolds relative to Lagrangian submanifolds)**.** *Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type $(k, r)$, $i : N \hookrightarrow M$ be a $k$-coisotropic submanifold satisfying*

$$W|_N \subseteq TN,$$

*and $L \hookrightarrow M$ be a $k$-Lagrangian submanifold complementary to $W$, that is, such that*

$$W|_L \oplus TL = TM|_L.$$

*Then there exists a neighborhood $U$ of $L$ in $M$, a submanifold $Q \hookrightarrow L$, a neighborhood $V$ of $L$ in $\bigwedge_r^k L$, and a multisymplectomorphism*

$$\phi : U \to V$$

*satisfying*

*i)* $\phi$ *is the identity on* $L$*, identified as the zero section in* $\bigwedge_r^k L$*;*

*ii)* $\phi(N \cap U) = \bigwedge_r^k L\big|_Q \cap V$.

*Proof.* Let $U, V$, and $\phi$ be the neighborhoods and multisymplectomorphism from Theorem 4.1.2 and define

$$Q := L \cap N.$$

We claim that

$$\phi(N \cap U) = \bigwedge_r^k L\big|_Q.$$

First recall that we have

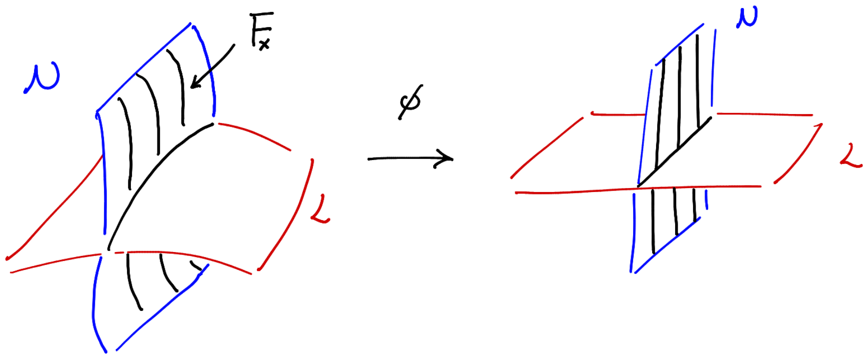
$$\phi_* W = W_L,$$

where $W_L$ is the canonical 1-Lagrangian distribution on $\bigwedge_r^k L$. Let $x \in L \cap N$ and $F_x$ be the leaf of $W$ through $x$. It is clear that $F_x \subseteq N$, and that, reducing $U$ and $V$ if necessary,

$$\phi(F_x \cap U) = \bigwedge_r^k T_x^* L \cap V,$$

since diffeomorphisms that preserve distributions preseve their leaves (when the distributions are integrable). Again, reducing $U$ and $V$ further, we may also assume that for every point $y \in N \cap U$ there is a point $x \in L \cap N$ such that the leaf of $W$ that contains $x$, $F_x$, also contains $y$, that is, we may assume that

$$N \cap U = \bigcup_{x \in L \cap N} F_x \cap U.$$




Therefore,

$$\phi(N \cap U) = \bigcup_{x \in L \cap N} \phi(F_x \cap U) = \bigcup_{x \in Q} \bigwedge_r^k T_x^* L \cap V = \bigwedge_r^k L\big|_Q,$$

proving the result. $\square$

**Theorem 4.1.4.** *Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type $(k, r)$, and $N \hookrightarrow M$ be a $k$-coisotropic submanifold satisfying*

$$W|_N \subseteq TN.$$

*Then, given any point $x \in N$, there exists a neighborhood $U$ of $x$ in $M$, a manifold $L$, a submanifold $Q \hookrightarrow L$, a neighborhood $V$ of $L$ in $\bigwedge_r^k L$ and a multisymplectomorphism*

$$\phi : U \to V$$

*such that*

*i) $\phi$ is the identity on $L$, idetified as the zero section in $\bigwedge_r^k L$;*

*ii) $\phi(N \cap U) = \bigwedge_r^k L\big|_Q \cap V$.*

*Proof.* Using Theorem 2.2.2, we can build a $k$-Lagrangian submanifold $L$ through any given point $x \in N$. Now the result follows using Theorem 4.1.3. □

## 4.2 Coisotropic reduction

When $k = 1$, that is, when $(M, \omega)$ is a symplectic manifold, recall that we have the classical result of coisotropic reduction due to Weinstein [Wei77].

**Theorem 4.2.1.** *[Coisotropic reduction in symplectic geometry] Let $(M, \omega)$ be a symplectic manifold, $i : N \hookrightarrow M$ be a coisotropic submanifold, and $j : L \hookrightarrow M$ be a Lagrangian submanifold that has clean intersection with $N$. Then, $TN^\perp$ is an integrable distribution and determines a foliation $\mathcal{F}$ of maximal integral leaves. Suppose that the quotient space $N/\mathcal{F}$ admits an smooth manifold structure such that the canonical projection*

$$\pi : N \to N/\mathcal{F}$$

*defines a submersion. Then there exists an unique symplectic form on $N/\mathcal{F}$, $\omega_N$ compatible with $\omega$ in the following sense*

$$\pi^* \omega_N = i^* \omega.$$

*Furthermore, if $\pi(N \cap L)$ is a submanifold, it is Lagrangian in $(N/\mathcal{F}, \omega_N)$.*

We would like to find an analogous result in multisymplectic manifolds. For the first part of Theorem 4.2.1, the classical argument works.

**Proposition 4.2.1** ([CIL99]). *Let $(M, \omega)$ be a multisymplectic manifold of order $k$ and $i : N \hookrightarrow M$ be a $k$-coisotropic submanifold. Then, $(TN)^{\perp,k} \cap TN \subseteq TN$ defines an involutive distribution.*

*Proof.* Let $X, Y \in \mathfrak{X}(N)$ be vector fields on $N$ with values in $(TN)^{\perp,k}$, and let $Z_1, \ldots, Z_k \in \mathfrak{X}(N)$ be arbitrary vector fields on $N$. Denote

$$\omega_0 := i^* \omega.$$

Since $\omega$ is closed, we have

$$
\begin{aligned}
0 = (d\omega_0)(X, Y, Z_1, \dots, Z_k) &= X(\omega_0(Y, Z_1, \dots, Z_k)) - Y(\omega_0(X, Z_1, \dots, Z_k)) \\
&+ \sum_{j=1}^{k} (-1)^j Z_i(\omega_0(X, Y, Z_1 \dots, \hat{Z}_i, \dots, Z_k)) - \omega_0([X, Y], Z_1, \dots, Z_k) \\
&+ \sum_{j=1}^{k} (-1)^{j+1} \omega_0([X, Z_i], Y, Z_1 \dots, \hat{Z}_i, \dots, Z_k) \\
&+ \sum_{j=1}^{k} (-1)^{j} \omega_0([Y, Z_i], X, Z_1 \dots, \hat{Z}_i, \dots, Z_k) \\
&+ \sum_{i<j} (-1)^{i+j} \omega_0([Z_i, Z_j], X, Y, Z_1 \dots, \hat{Z}_i, \dots, \hat{Z}_j, \dots, Z_k).
\end{aligned}
$$

Now, since both $X$ and $Y$ take values in $(TN)^{\perp,k}$, all the summands but

$$\omega_0([X, Y], Z_1, \dots, Z_k)$$

are zero. Therefore, we conclude

$$\omega_0([X, Y], Z_1, \dots, Z_k) = 0,$$

for all $Z_1, \dots, Z_k \in \mathfrak{X}(N)$, that is, $[X, Y]$ takes values in $(TN)^{\perp,k}$, proving that the distribution is involutive. □

If $(TN)^{\perp,k} \cap TN$ is regular, by Frobenius' Theorem, it determines a foliation $\mathcal{F}$ of maximal leaves. We have the following result.

**Theorem 4.2.2** ([CIL99]). *Let $(M, \omega)$ be a multisymplectic manifold of order k, and $i : N \hookrightarrow M$ be a k-coisotropic submanifold such that $(TN)^{\perp,k} \cap TN$ is regular. Suppose that $N/\mathcal{F}$ admits a smooth manifold structure such that the canonical projection*

$$\pi : N \to N/\mathcal{F}$$

*defines a submersion. Then there exists an unique multisymplectic form of order k on $N/\mathcal{F}$, $\omega_N$, that is compatible with $\omega$, that is,*

$$\pi^* \omega_N = i^* \omega.$$

*Proof.* Let $x \in N$. Notice that, since $\pi$ defines a submersion, we have the identification

$$T_{[x]} N/\mathcal{F} = T_x N / \ker d_x \pi = T_x N/(T_x N)^{\perp,k}.$$

Let $v_1, \dots, v_{k+1} \in T_x N$. The relation $\pi^* \omega_N = i^* \omega$ forces us to define

$$\omega_N|_{[x]}([v_1], \dots, [v_{k+1}]) := \omega|_x(v_1, \dots, v_{k+1}),$$

proving that $\omega_N$ is unique. It only remains to show that the previous definition does not depend on the choice of $x$ and $v_i$. For the latter, first observe that if $[v] = 0$, that is, $v \in (T_x N)^{\perp,k}$ we have

$$\omega(v, v_1, \dots, v_k) = 0,$$

for all $v_1, \dots, v_k \in T_xN$. Therefore, if $[v_i] = [u_i]$, for $i = 1, \dots, k+1$, we have

$$\begin{aligned}\omega_N|_{[x]}([v_1], \dots, [v_{k+1}]) &= \omega|_x(v_1, \dots, v_{k+1}) = \omega|_x(u_1, v_2, \dots, v_{k+1}) = \dots \\ &= \omega|_x(u_1, \dots, u_{k+1}) = \omega_N|_{[x]}([u_1], \dots, [u_{k+1}]).\end{aligned}$$

For the independence of the chosen point, given $x, y \in N$ in the same leaf, we can find a complete vector field $X$ on $N$ with values in $(TN)^{\perp,k}$ such that its flow satisfies

$$\phi_1^X(x) = y.$$

Now, denoting $\omega_0 := i^*\omega$, we have

$$£_X\omega_0 = \iota_X d\omega_0 + d\iota_X\omega_0 = 0,$$

since $\omega_0$ is closed and $\iota_X\omega_0 = 0$ (given that $X$ takes values in $(TN)^{\perp,k}$). This implies $(\phi_1^X)^*\omega_0 = \omega_0$. In particular, given $v_1, \dots, v_{k+1} \in T_xN$ we have

$$\begin{aligned}\omega_N|_{[x]}([v_1], \dots, [v_{k+1}]) &= \omega_0|_x(v_1, \dots, v_{k+1}) = \omega_0|_y(d_x\phi_1^X \cdot v_1, \dots, d_x\phi_1^X \cdot v_{k+1}) \\ &= \omega_N|_{[y]}([d_x\phi_1^X \cdot v_1], \dots, [d_x\phi_1^X \cdot v_{k+1}]).\end{aligned}$$

Since $X$ is tangent to $\mathcal{F}$, its flow $\phi_1^X$ leaves invariant the foliation, and $\pi \circ \phi = \pi$. In particular,

$$[v_i] = d_x\pi \cdot v_i = d_y\pi \cdot d_x\phi \cdot v_i = [d_x\phi \cdot v_i].$$

Finally, if $v_1, \dots, v_{k+1} \in T_xN$, $u_1, \dots, u_{k+1} \in T_yN$ with $[v_i] = [u_i]$,

$$\begin{aligned}\omega_N|_{[x]}([v_1], \dots, [v_{k+1}]) &= \omega_0|_x(v_1, \dots, v_{k+1}) = \omega_0|_y(d_x\phi_1^X \cdot v_1, \dots, d_x\phi_1^X \cdot v_{k+1}) \\ &= \omega_N|_{[y]}([d_x\phi_1^X \cdot v_1], \dots, [d_x\phi_1^X \cdot v_{k+1}]) \\ &= \omega_N|_{[y]}([u_1], \dots, [u_{k+1}]),\end{aligned}$$

proving the result. □

For the projection of Lagrangian submanifolds, the second part of Theorem 4.2.1, multisymplectic manifolds are *too general* and hard to study without asking for further structures. Indeed, we can easily find a counterexample.

**Example 4.2.1** (A counterexample). *Let $L = \langle l_1, l_2, l_3 \rangle$ be a 3-dimensional vector space and define*

$$V := L \oplus \bigwedge^2 V^*.$$

*Let $l^1, l^2, l^3$ be the dual basis induced on $L^*$ and denote*

$$\alpha^{ij} := l^i \wedge l^j.$$

*Then*

$$V = \langle l_1, l_2, l_3, \alpha^{12}, \alpha^{13}, \alpha^{23} \rangle.$$

*Let $l^1, l^2, l^3, \alpha_{12}, \alpha_{13}, \alpha_{23}$ be the dual basis. We have*

$$\Omega_L = \alpha_{12} \wedge l^1 \wedge l^2 + \alpha_{13} \wedge l^1 \wedge l^3 + \alpha_{23} \wedge l^2 \wedge l^3.$$

*Define*

$$N := \langle l_1 + l_2, l_1 + \alpha^{23}, l_2 + \alpha^{13}, l_3, \alpha^{12} \rangle.$$

*Then $N$ is a 2-coisotropic subspace. Indeed, a quick calcultion shows $N^{\perp,2} = 0$. This implies that the quotient space $N/N^{\perp,2}$ is (isomorphic to) $N$. Now, taking as the 2-Lagrangian subspace $L = \langle l_1, l_2, l_3 \rangle$, we have*

$$L \cap N = \langle l_1 + l_2, l_3 \rangle.$$

*However, this does not define a 2-Lagrangian subspace of $(N, \Omega_L|_N)$, since $\alpha^{12} \in (N \cap L)^{\perp,2}$, but $\alpha^{12} \notin (L \cap W)$.*

Nervertheless, we will be able to find a generalization of the previous theorem restricting the study to a particular class, those that locally are bundles of forms, which are precisely the multisymplectic manifolds of classical field theory [Got+04]. More particularly, we will study coisotropic reduction in multisymplectic manifolds of type $(k, r)$.

The classical proof of the last part of Theorem 4.2.1 uses en elaborate comparison of dimensions argument (see [AM08]). This argument hardly translates to multisymplectic manifolds since, in general, the map

$$TM \xrightarrow{\flat_1} \bigwedge^k M$$

does not define a bundle isomorphism. However, we can prove it using the local form proved in Section 4.1.

Given some manifold $L$, and a regular distribution on $L$, $\mathcal{E}$, define

$$M := \bigwedge_r^k L$$

endowed with its canonical multisymplectic structure. Here, the horizontal forms, are taken with respect to $\mathcal{E}$. Let $i : Q \hookrightarrow L$ be a submanifold of dimension at least $k$ (for $\bigwedge^k Q$ to be non-zero) and take

$$N := \bigwedge_r^k L\Big|_Q$$

the restricted bundle to $Q$. Then, $N \hookrightarrow M$ is a $k$-coisotropic submanifold. Indeed, under the (non-canonical) identification

$$T_{(x,\alpha)} N = T_x Q \oplus \bigwedge_r^k T_x^* L,$$

for $(x, \alpha) \in N$, we have

$$(TN)^{\perp,k} = 0 \oplus \ker i^*,$$

where $i^*$ is the induced map

$$i^* : \bigwedge_r^k T_x^*L \subseteq \bigwedge^k T_x^*L \to \bigwedge^k T_x^*Q.$$

We claim that the image of $\bigwedge_r^k T_x^*L$ under $i^*$ is $\bigwedge_r^k T_x^*Q$, where the horizontal forms are taken with respect to the subspace

$$\widetilde{\mathcal{E}_x} := \mathcal{E}_x \cap T_xQ.$$

Indeed, it is clear that

$$i^*\left(\bigwedge_r^k T_x^*L\right) \subseteq \bigwedge_r^k T_x^*Q,$$

since, if $e_1, \dots, e_r \in \widetilde{\mathcal{E}_x}$ and $\alpha \in \bigwedge_r^k T_x^*L$, we have

$$\iota_{e_1\wedge\cdots\wedge e_r} i^*\alpha = i^*(\iota_{e_1\wedge\cdots\wedge e_r}\alpha) = 0.$$

Now, to see the other inclusion, we take a projection

$$p : T_xL \to T_xQ$$

that satisfies $p(\mathcal{E}_x) = \widetilde{\mathcal{E}_x}$, that is, a projection that makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{E}_x & \hookrightarrow & T_xL \\ \big\downarrow {\scriptstyle p|_{\mathcal{E}_x}} & & \big\downarrow {\scriptstyle p} \\ \widetilde{\mathcal{E}_x} & \hookrightarrow & T_xQ \end{array} .$$

Take $\beta \in \bigwedge_r^k T_x^*Q$ and define $\alpha \in \bigwedge^k T_x^*L$ as

$$\alpha := p^*\beta.$$

It is clear that $i^*\alpha = \beta$. Furthermore, since $p$ satisfies $p(\mathcal{E}_x) = \widetilde{\mathcal{E}_x}$, we have

$$\alpha \in \bigwedge_r^k T_x^*L,$$

proving that

$$i^*\left(\bigwedge_r^k T_x^*L\right) = \bigwedge_r^k T_x^*Q.$$

In particular, when $\mathcal{E} \cap TQ$ has constant rank, so does[2] $(TN)^{\perp,k}$, and we have that the maximal integral leaf of this distribution that contains $(x, 0)$ is

$$\ker i^*|_{\{x\}} = \{(x, \alpha) : \alpha \in \ker i^*, \alpha \in \bigwedge_r^k T_x^*L\}.$$

[2] because $\operatorname{rank}(TN)^{\perp,k} = \operatorname{rank} \ker i^* = \operatorname{rank} \bigwedge_r^k L - \operatorname{rank} \bigwedge_r^k Q$

These leaves define a vector subbundle

$$\begin{array}{ccc} \ker i^* & \hookrightarrow & \bigwedge_r^k L\big|_Q \\ & \searrow \quad \swarrow & \\ & Q & \end{array} .$$

By the previous considerations, these bundles fit in a short exact sequence

$$0 \longrightarrow \ker i^* \hookrightarrow \bigwedge_r^k L \xrightarrow{i^*} \bigwedge_r^k Q \longrightarrow 0 \ .$$

Therefore, we may indentify

$$N/\mathcal{F} = \bigwedge_r^k Q,$$

where the horizontal forms are taken with respect to $\widetilde{\mathcal{E}} = \mathcal{E} \cap TQ$ (which we are assuming to have constant rank). A rutinary check shows that the multisymplectic structure induced from Theorem 4.2.2 is none other than the canonical multisymplectic structure on $\bigwedge_r^k Q$.

Now, let us study the projection of Lagrangian submanifolds. An important class of $k$-Lagrangian submanifolds in $\bigwedge_r^k L$ are given by closed forms (Proposition 2.2.1)

$$\alpha : L \to \bigwedge_r^k L.$$

We have the following diagram

$$\begin{array}{ccc} L & \xrightarrow{\alpha} & \bigwedge_r^k L = M \\ \uparrow & & \uparrow \\ Q & \xrightarrow{\alpha|_Q} & \bigwedge_r^k L\big|_Q = N \\ & \overset{i^*\alpha}{\searrow} & \downarrow i^* = \pi \\ & & \bigwedge_r^k Q = N/\mathcal{F} \end{array} .$$

It is clear that the projection of $\alpha(L) \cap N$ onto $N/\mathcal{F} = \bigwedge_r^k Q$ is exactly the image of

$$i^*\alpha : Q \to \bigwedge_r^k Q.$$

Since $\alpha$ is closed, so is $i^*\alpha$, proving that in this local form, $k$-Lagrangian submanifolds complementary to the vertical distribution $\mathcal{W}$ reduce to $k$-Lagrangian submanifolds. Therefore, using Theorem 4.1.3 we have the main result of this section:

**Theorem 4.2.3.** *Let $(M, \omega, \mathcal{W}, \mathcal{E})$ be a multisymplectic manifold of type $(k, r)$, $i : N \hookrightarrow M$ a $k$-coisotropic submanifold satisfying*

$$\mathcal{W}\big|_N \subseteq TN,$$

*and $j : L \hookrightarrow M$ a $k$-Lagrangian submanifold complementary to $\mathcal{W}$. Suppose that $N/\mathcal{F}$ admits a smooth manifold strcuture such that $\pi : N \to N/\mathcal{F}$ defines a submersion, where $\mathcal{F}$ is the foliation associated to $(TN)^{\perp,k}$ (see Theorem 4.2.2), and that*

$$\mathcal{E}\big|_N \cap \left(TN/\mathcal{W}\big|_N\right)$$

*has constant rank. Then, if $\pi(L \cap N)$ is a submanifold, it is $k$-Lagrangian .*

# Conclusions

In this text we have analysed the role that Lagrangian and coisotropic submanifolds play in multisymplectic geometry, with the intention of extending as far as possible the well-known results in symplectic geometry. When dealing with forms of degree higher than 2, we have seen that there are different complements to a submanifold, which enriches the geometry but at the same time makes it more complex. One of the first results obtained is the interpretation of Lagrangian submanifolds as possible dynamics in Chapter 3, as well as the introduction of a graded bracket algebra. This makes it possible to deal with currents and conserved quantities. The main result of the paper is a coisotropic reduction theorem (Chapter 4) which we hope will be useful in applications to multisymplectic field theory.

In future work we have proposed the following objectives:

1. Apply the results obtained in the current paper to multisymplectic field theories.
2. Since some field theories are singular, we would like to develop a regularization method as in the case of singular Lagrangian dynamics (see [IM95]); previously, we have to prove a coisotropic embedding theorem á la Gotay [Got82; SZ17] in the context of multisymplectic geometry.
3. Develop the covariant approach through a space-time decomposition, and interpret the coisotropic reduction in the corresponding infinite dimensional setting.
4. Extend the results to the realm of multicontact geometry (see [Leó+23]).

# Appendices

# Appendix A

# A crash course on Symplectic Geometry

For a more in depth treatment of symplectic geometry and its connection to classical mechanics we refer to [AM08; Arn78].

## A.1 Symplectic vector spaces

**Definition A.1.1** (Symplectic vector space)**.** A **symplectic vector space** is a pair $(V, \omega)$, where $V$ is a $2n$-dimensional vector space, and $\omega$ is a non-degenerate 2-form on $V$, where non-degeneracy means that the map

$$V \to V^*;\ v \mapsto \iota_v \omega$$

defines a linear isomorphism.

Notice that the requirement for $V$ to be even-dimensional is necessary, since an antisymmetric map has even rank.

**Definition A.1.2** (Symplectomorhism)**.** Let $(V_1, \omega_1)$, $(V_2, \omega_2)$ be symplectic vector spaces. A symplectomorphism is a linear map

$$f : V_1 \to V_2$$

satisfying

$$f^* \omega_2 = \omega_1.$$

**Definition A.1.3.** Orthogonal Let $(V, \omega)$ be a symplectic vector space and $W \subseteq V$ a subspace. Define the **symplectic orthogonal** of $W$ as

$$W^\perp := \{v \in V :\ \iota_v \omega|_W = 0\}.$$

Since $\flat : V \to V^*$ defines a linear isomorphism and

$$W^\perp = \ker i^* \circ \flat,$$

where $i^* : V^* \to W^*$ denotes the restriction of linear forms, we have

$$\dim W^\perp = 2n - \dim W.$$

**Definition A.1.4.** A subspace $W$ of a symplectic vector space is called

(i) **isotropic**, if $W \subseteq W^{\perp}$;

(ii) **coisotropic**, if $W^{\perp} \subseteq W$;

(iii) **Lagrangian**, if $W^{\perp} = W$;

(iv) **symplectic**, if $W \cap W^{\perp} = 0$.

**Remark A.1.1.** Notice that the equality $\dim W + \dim W^{\perp} = 2n$ imply that for isotropic $W$, $\dim W \leq n$; for coisotropic $W$, $\dim W \geq n$; and for Lagrangian $W$, $\dim W = n$. In particular, Lagrangian subspace are minimally coisotropic and maximally isotropic.

One would like to characterize symplectic vector spaces up to the notion of isomorphism defined above. Fortunately, there is an unique vector space for each dimension:

**Proposition A.1.1.** *Let $(V, \omega)$ be a symplectic manifold of dimension $2n$. Then, there exists a basis $(x_1, \dots, x_n, y^1, \dots, y^n)$ such that*

$$\omega = x^i \wedge y_i,$$

*where $x^1, \dots, x^n, y_1, \dots, y_n$ denotes the dual basis.*

*Proof.* Fix an arbitrary vector $x_1 \in V$. Then, since $\omega$ is non-degenerate, there is a vector $y^1 \in V$ such that

$$\omega(x_1, y^1) = 1.$$

Let

$$W := \langle x_1, y^1 \rangle.$$

It is easy to see that $W^{\perp} \cap W = 0$ and, therefore,

$$W \oplus W^{\perp} = V.$$

Iterating this argument on $W^{\perp}$ and so on, we obtain the result. □

As a corollary, every symplectic vector space is characterized by its dimension.

## A.2 Symplectic manifolds

**Definition A.2.1** (Symplectic manifold)**.** A **symplectic manifold** is a pair $(M, \omega)$, where $M$ is a manifold and $\omega \in \Omega^2(M)$ is a closed, non-degenerate 2-form.

The definitions of isotorpic, coisotropic, Lagrangian, sympelctic extend naturally to submanifolds as follows:

**Definition A.2.2.** A submanifold $i : N \hookrightarrow M$ is called isotropic (re. coisotropic, Lagrangian, symplectic) if $T_qN \subseteq T_qM$ is isotropic (res. coisotropic, Lagrangian, symplectic), for every $q \in N$.

**Observation A.2.1.** Notice that every symplectic manifold is necessarily even dimensional, since its tanget space should be.

The isomorphism between symplectic manifolds is given as follows:

**Definition A.2.3** (Symplectomorphism)**.** A **symplectomorphism** between two symplectic manifolds $(M, \omega_1)$, $(M, \omega_2)$ is a diffeomorphism $f : M_1 \to M_2$ satisfying

$$f^*\omega_2 = \omega_1.$$

**Example A.2.1** (The cotangent bundle)**.** *The main example of symplectic manifolds is the phase space of a mechanical system. More specifically, let Q be a manifold (which usually represents the configuration space of a mechanical system). Then the phase space is $T^*Q$, the cotangent bundle. In $T^*Q$ there is a tautological 1-form, the* ***Liouville 1-form*** *defined as*

$$\theta_Q\Big|_\alpha(v) := \alpha(\pi_* v),$$

*where $\alpha \in T^*Q$, $v \in T_\alpha T^*Q$, and $\pi : T^*Q \to Q$ denotes the canonical proyection. In canonical coordinates $(q^i, p_i)$,*

$$\theta_Q = p_i dq^i,$$

*where we are making use of the Einstein sumation convention, which will be used throughout the rest of the text. If we define*

$$\omega_Q := -d\theta_Q = dq^i \wedge dp_i,$$

*$(T^*Q, \omega)$ is a symplectic manifold.*

**Proposition A.2.1.** *For every 1-form $\alpha : Q \to T^*$ we have*

$$\alpha^*\theta_Q = \alpha, \ \ \alpha^*\omega_Q = -d\alpha,$$

*and $\theta_Q$, $\omega_Q$ are the unique forms satisfying this.*

*Proof.* Since $\omega_Q = -d\theta_Q$, it is enough to prove $\alpha_Q^\theta = \alpha$. Indeed, let $x \in Q$ and $v \in T_x Q$, then

$$\langle (\alpha^*\theta_Q)_x, v\rangle = \langle \theta_Q|_{\alpha_x}, \alpha_* v\rangle = \alpha_x(\pi_*\alpha_* v) = \alpha_x(v).$$

Uniqueness is easily checked in coordinates or using Lemma 3.3.1. □

Example A.2.1 is *The example* of symplectic manifolds, in the sense that every symplectic manifold is locally symplectomorphic to a cotangent bundle.

**Theorem A.2.1** (Darboux Theorem)**.** *Let $(M, \omega)$ be a symplectic manifold, and let $x \in M$. Then there exists a neighborhood $U$ of $x$ in $M$, a manifold $Q$, an open subset $V \subset T^*Q$, and a symplectomorphism $f : U \to V$.*

**Observation A.2.2.** Given a symplectic manifold $(M, \omega)$, Theorem A.2.1 together with the local expression of $\omega_Q$ in Example A.2.1 imply that around any point we can find coordinates $(q^i, p_i)$ such that

$$\omega = dq^i \wedge dp_i.$$

These type of coordinates are called **Darboux coordinates**.

## A.3 Hamiltonian vector fields

As we have mentioned, symplectic manifolds give a geometric framework to Classical Mechanics. In this formalism, dynamics are modeled by Hamiltonian vector fields, which we define below.

**Definition A.3.1** (Hamiltonian vector field)**.** Let $(M, \omega)$ be a symplectic manifold and $H \in C^\infty(M)$ be a function, which will be called the Hamiltonian. Then, the vector field $X_H \in \mathfrak{X}(M)$ satisfying

$$\iota_{X_H}\omega = dH$$

will be called **the Hamiltonian vector field** of $H$.

**Observation A.3.1.** Notice that a vector field $X \in \mathfrak{X}(M)$ is a Hamiltonian vector field if and only if $\iota_X\omega$ is exact. If it were closed, $X$ is called **locally Hamiltonian**.

**Observation A.3.2.** In Darboux coordinates, Hamiltonian vector fields have the expression

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

Notice that the equations of motion $X_H$ defines are

$$\dot{q}^i = \frac{\partial H}{\partial p_i},$$
$$\dot{p}_i = \frac{\partial H}{\partial q^i},$$

which are Hamilton's equations of motion.

A Hamiltonian flow, meaning the flow of a Hamiltonian vector field satisfyies the following propoerties:

**Theorem A.3.1** (Liouville)**.** *Denote by $\varphi_t$ the flow of a Hamiltonian vector field $X_H$. Then, $\varphi_t^*\omega = \omega$, for every $t$. In particular, it preserves the sympelctic volume form $\omega^n$.*

*Proof.* Indeed,

$$\pounds_{X_H}\omega = d\iota_{X_H}\omega = d^2H = 0.$$

□

**Theorem A.3.2** (Conservation of Energy)**.** *$H$ is constant along the integral curves of $X_H$.*

*Proof.* It follows from the equality

$$X_H(H) = \omega(X_H, X_H) = 0.$$

□

## A.4 The Poisson bracket

**Definition A.4.1** (Poisson bracket)**.** Given a symplectic manifold $(M, \omega)$ and two functions $f, g \in C^\infty(M)$, define the Poisson bracket as

$$\{f, g\} := \omega(X_f, X_g).$$

**Proposition A.4.1.** *We have*

$$[X_f, X_g] = -X_{\{f,g\}}.$$

*Proof.* If follows from

$$\iota_{[X_f,X_g]}\omega = £_{X_f}\iota_{X_g}\omega - \iota_{X_g}£_{X_f}\omega = -\{f, g\}.$$

□

**Proposition A.4.2.** *The Poisson bracket defines a Lie algebra structure on $C^\infty(M)$.*

*Proof.* It suffices to prove that the Jacobi identity holds. Indeed, for $f, g, h \in C^\infty(M)$, we have

$$0 = (d\omega)(X_f, X_g, X_h) = 2\left(\{\{f, g\}, h\} + \text{cycl.}\right),$$

where we have used the formula

$$\begin{aligned}(d\alpha)(X_1, \dots, X_{r+1}) = &\sum_{i=1}^{r+1}(-1)^{i+1}X_i(\alpha(X_1, \dots, \hat{X}_i, \dots, X_{r+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{r+1}),\end{aligned}$$

for $\alpha \in \Omega^r(M)$, and the previous proposition. □

**Proposition A.4.3.** *A submanifold $i : N \hookrightarrow M$ is coisotropic if and only if, for every couple of functions $f, g$ (locally) constant on $N$, $\{f, g\}$ vanishes on $N$.*

*Proof.* We can prove it locally. Therefore, we can assume that $N = \{\phi_\alpha = 0\}$, for certain functionally independent functions $\Phi_\alpha$. Then, $N$ is coisotropic if and only if

$$(TN)^\perp = \langle X_{\Phi_\beta}\rangle \subseteq TN = \bigcap_\alpha \ker d\Phi_\alpha,$$

that is, if and only if

$$0 = d\Phi_\alpha \cdot X_{\Phi_\beta} = \{\Phi_\alpha, \Phi_\beta\},$$

for every $\alpha, \beta$. Since every function $f$ that is constant on $N$ can be expressed as

$$f = f^\alpha \Phi_\alpha + C,$$

for some functions $f^\alpha$ and $C \in \mathbb{R}$, it follows that $\{\Phi_\alpha, \Phi_\beta\} = 0$ is equivalent to what we wanted to prove. □

# Appendix B

# The first Jet Bundle

In this appendix we formally secify what we mean by the firt Jet Bundle and we define the vector valued form $S_\omega$ used in Chapter 1. For a more in-depth treatment of Jet Bundles, we refer to [Sau89].

## B.1 The formal definition

Let $\pi : Y \to X$ be a fibered manifold.

**Definition B.1.1.** Define an equivalence relation on the set of local sections as follows. Let $\phi : U \to Y, \psi : V \to Y$ be local sections, where $U, V$ are open subsets of $X$. For $x \in U \cap V$, we write

$$\phi \sim_x \psi,$$

if $\phi(x) = \psi(x)$ and $d_x\phi = d_x\psi$. Denote by

$$j^1_x\phi$$

the equivalence class of $\phi$ modulo the relation $\sim_x$ .

**Definition B.1.2** (The first Jet Bundle)**.** The **first Jet Bundle** (as a set) is

$$J^1\pi := \{j^1_x\phi : \ \phi \text{ local section of } \pi\}.$$

We can endow this set with an smooth manifold structure with the following atlas. For each set of fibered coordintes on $\pi : Y \to X$, $(x^\mu, y^i)$ defined on an open set $U$ of $\mathbb{R}^{n+m}$ $(\dim X = n, \dim Y = m)$, define the map

$$X : U \times \mathbb{R}^{nm} \to J^1\pi$$

as

$$X(x^\mu, y^i, z^i_\mu) := j^1_x\phi,$$

where $\phi$ is the local section (defined in coordinates)

$$\phi^i = z^i_\mu x^\mu,$$

and $x \in X$ is the point represented by the coordinates $(x^\mu)$. Denote by $\mathcal{A}$ the atlas defined by the previous maps.

**Proposition B.1.1.** *$\mathcal{A}$ is a smooth atlas and, thus, $J^1\pi$ is a $(n+m+nm)$−dimensional smooth manifold.*

*Proof.* Let $(x^\mu, y^i)$, $(\widetilde{x}^\nu, \widetilde{y}^j)$ be fibered coordinates defined on the same open subset of $Y$. Denote by $(x^\mu, y^i, z^i_\mu)$, $(\widetilde{x}^\nu, \widetilde{y}^j, \widetilde{z}^j_\nu)$ the induced maps on $J^1\pi$, respectively. The first set of coordinates asigns the section $\phi^i = z^i_\mu x^\mu$. Now, expressing this section locally in the second set of coordinates $\widetilde{\phi}^j := \widetilde{y}^j \circ \phi$, we have

$$\frac{\partial \widetilde{\phi}^j}{\partial \widetilde{x}^\nu} = \frac{\partial \widetilde{y}^j}{\partial x^\mu}\frac{\partial x^\mu}{\partial \widetilde{x}^\nu} + \frac{\partial \widetilde{y}^j}{\partial y^i}\frac{\partial \phi^i}{\partial x^\mu}\frac{\partial x^\mu}{\partial \widetilde{x}^\nu}$$

and, therefore, the change of coordinates is

$$\widetilde{z}^j_\nu = \frac{\partial \widetilde{y}^j}{\partial x^\mu}\frac{\partial x^\mu}{\partial \widetilde{x}^\nu} + \frac{\partial \widetilde{y}^j}{\partial y^i}\frac{\partial x^\mu}{\partial \widetilde{x}^\nu} z^i_\mu,$$

which is differentiable. Repeating the same argument we show that inverse change of coordinates is also differentiable and thus, $\mathcal{A}$ is a smooth atlas. □

Writing the previous change of coordinates as

$$\widetilde{z}^j_\nu = a^j_\nu + b^{j\mu}_{\nu i} z^i_\mu,$$

we notice that this endows $J^1\pi$ with an affine bundle structure over $Y$. Intrinsically, for each $y \in Y$, we can identify each $j^1_x\phi$ satisfying $\phi(x) = y$ with the section of $d_y\pi$,

$$d_x\phi : T_xX \to T_yY.$$

Therefore, any couple of elements $j^1_x\phi, j^1_x\psi \in J^1\pi$ with $\phi(x) = \psi(x) = y$ define a map

$$d_x\phi - d_x\psi : T_xX \to \ker d_y\pi$$

or, equivalently, an element of

$$d_x\phi - d_x\psi \in T^*_xX \otimes \ker d_y\pi.$$

It is easy to see that this defines an affine structure on $J^1_y\pi$ (here $J^1_y\pi$ denotes the $y$-fiber) modeled on the vector space $T^*_xX \otimes \ker d_y\pi$. Extending this construction globally, we conclude that $J^1\pi \to Y$ is an affine bundle modeled on the vector bundle

$$\pi^*(T^*X) \otimes \ker d\pi \to Y.$$

## B.2 The vertical lift

We can use the affine structure defined on $J^1\pi \to Y$ to define a "lift" as follows.

Notice that we can identify the vertical subspace of $T_{j^1_x\phi}J^1\pi$ with

$$\mathcal{V}_{j^1_x\phi} = T^*_xX \otimes \ker d_{\phi(x)}\pi.$$

Therefore, if we fix a point $j^1_x\phi \in J^1\pi$, we can assign to each element $T^*_xX \otimes \ker d_\phi(x)\pi$ a vertical vector. In coordinates, this takes the expression

$$dx^\mu \otimes \frac{\partial}{\partial y^i} \mapsto \frac{\partial}{\partial z^i_\mu}$$

which we can express alternatively as

$$\frac{\partial}{\partial x^\mu} \otimes dy^i \otimes \frac{\partial}{\partial z^i_\mu}.$$

Notice that this *does not* defines a global tensor on $J^1\pi$. To obtain a global tensor, hopefully (we will shortly see that there is no obvious way of defining it) of type $(2,1)$, given $j^1_x\phi$, we need a canonical way of obtaining a form in $T^*_xX$ and a vector in $\ker d_y\pi$ from a form and a vector in $T^*_{j^1\phi}J^1\pi$ and $T_{j^1\phi}J^1\pi$, respectively. Since we cannot project a form onto $T^*_xX$ (this is the main issue), let us focus on obtaning a vector in $\ker d_y\pi$.

A vector $\xi \in T_{j^1_x\phi}J^1\pi$ projects onto a vector $\pi_*\xi \in T_yY$. How can we obtain an element of $\ker d_y\pi$? To do so, we need a decomposition

$$T_yY = \ker d_y\pi \oplus H.$$

Usually, there is no canonical way of obtaining $H$ but, since we have $j^1_x\phi$ fixed, we might as well define

$$H := d_x\phi\,(T_xX)\,.$$

In coordinates,

$$\ker d_y\pi = \left\langle \frac{\partial}{\partial y^i} \right\rangle,\ H = \left\langle \frac{\partial}{\partial x^\nu} + z^i_\nu \frac{\partial}{\partial y^i} \right\rangle.$$

Now, we have a well defined element of $\ker d_y\pi$, the projection of $\pi_*\xi$ under the previopus decompositon. Locally, this map is

$$(dy^i - z^i_\nu dx^\nu) \otimes \frac{\partial}{\partial y^i}.$$

Since we cannot obtain an element $\alpha \in T^*_xX$, if we fix $\alpha \in \Omega^1(X)$,

$$\alpha = \alpha_\mu \otimes dx^\mu,$$

then we get a well-defined tensor

$$\alpha_\mu (dy^i - z^i_\nu dx^\nu) \otimes \frac{\partial}{\partial z^i_\mu}.$$

An alternative way of obtaining a 1-form on $X$ is via a volume form $\omega$ on $X$. Identifying $\omega$ with its pull-back to $J^1\pi$, each set of $n-1$ vectors $\xi_1, \dots, \xi_{n-1} \in T_{j^1_x\phi}J^1\pi$ ($n = \dim X$) defines a semi-basic form $\iota_{\xi_1\wedge\cdots\wedge\xi_{n-1}}\omega$ which is the pull-back of certain form (which we denote the same)

$\iota_{\xi_1\wedge\cdots\wedge\xi_{n-1}}\omega \in T_x^*X$. Coupling this with the above tensor and changing the ordering, we obtain a tensor with the expression

$$\widetilde{S}_\omega = (dy^i - z_\nu^i) \otimes d^{n-1}x_\mu \otimes \frac{\partial}{\partial z_\mu^i},$$

where $x^\mu$ are coordinates on $X$ satisfying $\omega = d^n x$. For applications, it is more interesting to have a vector valued $n$-form and, therefore, we define **the vertical lift** as the atisymmetrization of $\widetilde{S}_\omega$, which locally reads

$$S_\omega = (dy^i - z_\nu^i) \wedge d^{n-1}x_\mu \otimes \frac{\partial}{\partial z_\mu^i}.$$

# Bibliography


[AM08] R. Abraham and J. Marsden. *Foundations of Mechanics*. American Mathematical Society, May 2008. ISBN: 9781470411343. DOI: 10.1090/chel/364. URL: http://dx.doi.org/10.1090/chel/364.

[Arn78] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer New York, 1978. ISBN: 9781475716931. DOI: 10.1007/978-1-4757-1693-1. URL: http://dx.doi.org/10.1007/978-1-4757-1693-1.

[Bla21] C. Blacker. “Reduction of multisymplectic manifolds”. In: *Letters in Mathematical Physics* 111.3 (May 2021). ISSN: 1573-0530. DOI: 10.1007/s11005-021-01408-y. URL: http://dx.doi.org/10.1007/s11005-021-01408-y.

[BSF88] E. Binz, J. Sniatycki, and H. Fischer. *Geometry of Classical Fields*. Elsevier, 1988. ISBN: 9780444705440. DOI: 10.1016/s0304-0208(08)x7101-4. URL: http://dx.doi.org/10.1016/s0304-0208(08)x7101-4.

[Car22] E. Cartan. *Leçons sur les invariants intégraux*. Hermann, 1922.

[Car33] E. Cartan. “Les espaces métriques fondés sur la notion d’aire”. In: *Actualités Sc. et Industr.* 72 (1933).

[CIL96] F. Cantrijn, A. Ibort, and M. de León. “Hamiltonian structures on multisymplectic manifolds”. In: *Rend. Sem. Mat. Univ. Politec. Torino* 54.3 (Jan. 1996), pp. 225–236.

[CIL99] F. Cantrijn, A. Ibort, and M. de León. “On the geometry of multisymplectic manifolds”. In: *Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics* 66.3 (1999), pp. 303–330. DOI: 10.1017/S1446788700036636.

[DeD29] DeDonder. *Théorie invariantive du calcul des variations*. Bull. Acad. de Belg., 1929.

[Dir58] Paul Adrien Maurice Dirac. “Generalized Hamiltonian dynamics”. In: *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 246 (1958), pp. 326–332. URL: https://api.semanticscholar.org/CorpusID:119748805.

[FPR03] M. Forger, C. Paufler, and H. Romer. “The Poisson bracket for Poisson forms in multisymplectic theory”. In: *Reviews in Mathematical Physics* 15.07 (Sept. 2003), pp. 705–743. ISSN: 1793-6659. DOI: 10.1142/s0129055x03001734. URL: http://dx.doi.org/10.1142/s0129055x03001734.

[Got+04] M.J. Gotay et al. *Momentum Maps and Classical Relativistic Fields. Part I: Covariant Field Theory*. 2004. arXiv: physics/9801019 [math-ph].

[Got82] M.J. Gotay. “On coisotropic imbeddings of presymplectic manifolds”. In: *Proc. Amer. Math. Soc.* 84.1 (1982), pp. 111–114. DOI: 10.1090/S0002-9939-1982-0633290-X.

[Got88] M.J. Gotay. “A multisymplectic approach to the KdV equation”. In: *Differential geometrical methods in theoretical physics (Como, 1987)*. Vol. 250. NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. Kluwer Acad. Publ., Dordrecht, 1988, pp. 295–305. ISBN: 90-277-2820-8. DOI: 10.1007/978-94-015-7809-7_15.

[Got91a] M.J. Gotay. “A multisymplectic framework for classical field theory and the calculus of variations. I. Covariant Hamiltonian formalism”. In: *Mechanics, analysis and geometry: 200 years after Lagrange*. North-Holland Delta Ser. North-Holland, Amsterdam, 1991, pp. 203–235. ISBN: 0-444-88958-2. DOI: 10.1016/B978-0-444-88958-4.50012-4.

[Got91b] M.J. Gotay. “A multisymplectic framework for classical field theory and the calculus of variations. II. Space + time decomposition”. In: *Differential Geom. Appl.* 1.4 (1991), pp. 375–390. ISSN: 0926-2245,1872-6984. DOI: 10.1016/0926-2245(91)90014-Z. URL: https://doi.org/10.1016/0926-2245(91)90014-Z.

[IM95] A Ibort and J. Marín-Solano. “Coisotropic regularization of singular Lagrangians”. In: *J. Math. Phys.* 36.10 (1995), pp. 5522–5539.

[LDS03] M. de Leon, D. Martín de Diego, and A. Santamaria-Merino. “Tulczyjew’s triples and lagrangian submanifolds in classical field theories”. In: *Applied Differential Geometry and Mechanics*. 2003, pp. 21–47.

[Leó+23] M. de León et al. “Multicontact formulation for non-conservative field theories”. In: *J. Phys. A* 56.2 (2023), No. 025201, 44 pp.

[LMS04] Manuel de León, David Martín de Diego, and Aitor Santamaría-Merino. “Symmetries in classical field theory”. In: *Int. J. Geom. Methods Mod. Phys.* 1.5 (2004), pp. 651–710. ISSN: 0219-8878,1793-6977. DOI: 10.1142/S0219887804000290. URL: https://doi.org/10.1142/S0219887804000290.

[LR89] M. de León and P.R. Rodrigues. *Methods of differential geometry in analytical mechanics*. North-Holland, Amsterdam, 1989.

[Mar88] G. Martin. “A Darboux theorem for multi-symplectic manifolds”. In: *Letters in Mathematical Physics* 16.2 (Aug. 1988), pp. 133–138. ISSN: 1573-0530. DOI: 10.1007/bf00402020. URL: http://dx.doi.org/10.1007/bf00402020.

[Rom09a] N. Román-Roy. “Multisymplectic Lagrangian and Hamiltonian Formalisms of Classical Field Theories”. In: *Symmetry, Integrability and Geometry: Methods and Applications* (Nov. 2009). ISSN: 1815-0659. DOI: 10.3842/sigma.2009.100. URL: http://dx.doi.org/10.3842/sigma.2009.100.

[Rom09b] Narciso Román-Roy. “Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories”. In: *SIGMA Symmetry Integrability Geom. Methods Appl.* 5 (2009), Paper 100, 25. ISSN: 1815-0659. DOI: 10.3842/SIGMA.2009.100. URL: https://doi.org/10.3842/SIGMA.2009.100.

[RW15] L. Ryvkin and T. Wurzbacher. “Existence and unicity of co-moments in multisymplectic geometry”. In: *Differential Geom. Appl.* 41 (2015), pp. 1–11. ISSN: 0926-2245,1872-6984. DOI: 10.1016/j.difgeo.2015.04.001. URL: https://doi.org/10.1016/j.difgeo.2015.04.001.

[RW19] L. Ryvkin and T. Wurzbacher. “An invitation to multisymplectic geometry”. In: *Journal of Geometry and Physics* 142 (2019), pp. 9–36. ISSN: 0393-0440. DOI: https://doi.org/10.1016/j.geomphys.2019.03.006. URL: https://www.sciencedirect.com/science/article/pii/S0393044018301189.

[RWZ20] L. Ryvkin, T. Wurzbacher, and M. Zambon. “Conserved quantities on multisymplectic manifolds”. In: *J. Aust. Math. Soc.* 108.1 (2020), pp. 120–144. ISSN: 1446-7887,1446-8107. DOI: 10.1017/s1446788718000381. URL: https://doi.org/10.1017/s1446788718000381.

[Sau89] D. J. Saunders. *The Geometry of Jet Bundles*. Cambridge University Press, Mar. 1989. ISBN: 9780511526411. DOI: 10.1017/cbo9780511526411. URL: http://dx.doi.org/10.1017/cbo9780511526411.

[SW19] G. Sevestre and T. Wurzbacher. “Lagrangian Submanifolds of Standard Multisymplectic Manifolds”. In: *Geometric and Harmonic Analysis on Homogeneous Spaces*. Springer International Publishing, 2019, pp. 191–205. ISBN: 9783030265625. DOI: 10.1007/978-3-030-26562-5_8. URL: http://dx.doi.org/10.1007/978-3-030-26562-5_8.

[SZ17] F. Schätz and M. Zambon. “Equivalences of coisotropic submanifolds”. In: *J. Symplectic Geom.* 15.1 (2017), pp. 107–149.

[Vai94] I. Vaisman. *Lectures on the Geometry of Poisson Manifolds*. Birkhäuser Basel, 1994. ISBN: 9783034884952. DOI: 10.1007/978-3-0348-8495-2. URL: http://dx.doi.org/10.1007/978-3-0348-8495-2.

[Wei71] A. Weinstein. “Symplectic manifolds and their Lagrangian submanifolds”. In: *Advances in Mathematics* 6.3 (1971), pp. 329–346. ISSN: 0001-8708. DOI: https://doi.org/10.1016/0001-8708(71)90020-X. URL: https://www.sciencedirect.com/science/article/pii/000187087190020X.

[Wei77] A. Weinstein. *Lectures on Symplectic Manifolds*. American Mathematical Society, Dec. 1977. ISBN: 9781470423896. DOI: 10.1090/cbms/029. URL: http://dx.doi.org/10.1090/cbms/029.